Semiclassical measure for the solution of the dissipative Helmholtz equation

Julien Royer

Abstract

We study the semiclassical measures for the solution of a dissipative Helmholtz equation with a source term concentrated on a bounded submanifold. The potential is not assumed to be non-trapping, but trapped trajectories have to go through the region where the absorption coefficient is positive. In that case, the solution is microlocally written around any point away from the source as a sum (finite or infinite) of lagragian distributions. Moreover we prove and use the fact that the outgoing solution of the dissipative Helmholtz equation is microlocally zero in the incoming region.

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1 Introduction and statement of the result

We consider on $L^2(\mathbb{R}^n)$ the dissipative semiclassical Helmholtz equation:

$$(-h^2\Delta + V_h - E_h)u_h = S_h \tag{1.1}$$

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in the high frequency limit, that is when the semiclassical parameter $h > 0$ goes to 0. Here the potential $V_h = V_1 - ihV_2$ has a nonpositive imaginary part of size h. We recall (see [BLSS03]) that this equation modelizes for instance the propagation of the electromagnetic field of a laser in material medium. In this setting the parameter h is the wave length of the laser, $\mathrm{Re}(E_h - V_h)$ is linked to the electronic density of the material medium (and plays the role of the refraction index for the corresponding hamiltonian problem) while $h^{-1}\,\mathrm{Im}(E_h - V_h)$ is the absorption coefficient of the laser energy by the material.

Thus, in order to consider the case of a non-constant absorption coefficient we have to allow non-real potentials. We proved in [Roy] that if the potential has non-positive imaginary part then (with decay and regularity assumptions on V) the resolvent $(-h^2\Delta + V_h - z)^{-1}$ is well-defined for $\mathrm{Im}\, z > 0$ and is of size $O(h^{-1})$ uniformly for z close to $E \in \mathbb{R}_+^*$ on condition E satisfies an assumption on classical trajectories for the corresponding hamiltonian problem. In this case, the resolvent has a limit for $z \to E$ in the space of bounded operators in some weighted spaces, and this limit operator gives the (outgoing) solution for (1.1) (see below).

Given a source term S_h and such an energy $E > 0$, our purpose in this paper is to study the asymptotic when $h \to 0$ for the outgoing solution u_h of (1.1). More precisely we are interested in the semiclassical measures (or Wigner measures) of u_h. The first work in this direction seems to be the paper of J.-D. Benamou, F. Castella, T. Katsaounis and B. Perthame ([BCKP02]). In their paper $S_h = S(x/h)/h$ concentrates on 0 and $\mathrm{Im}\, E_h = h\alpha_h$ with $\alpha_h \to \alpha \geqslant 0$. They consider the family of Wigner transforms f_h of the solutions u_h and prove that after extracting a subsequence, this family of Wigner transforms converges to a measure f which is the (outgoing) solution of the transport equation[1]:

$$\alpha f + \xi.\partial_x f(x,\xi) - \frac{1}{2}\partial_x V_1(x).\partial_\xi f(x,\xi) = \frac{1}{(4\pi)^2}\delta(x)\left|\hat{S}(\xi)\right|^2 \delta(|\xi| = 1) \tag{1.2}$$

Note that the solution is estimated by Morrey-Companato-type estimates (see [PV99]) and that part of the result is left as a conjecture and proved in [Cas05].

F. Castella, B. Perthame and O. Runborg study in [CPR02] the similar problem with a source term which concentrates on an unbounded submanifold of $\mathbb{R}^n$. As a consequence there is a lack of decay of the source and Morrey-Companato estimates cannot be used. Actually only a formal description of the asymptotics is given and the proof concerns the case where the refraction index is constant, that is $V_1 = 0$, and the submanifold is an affine subspace. X.-P. Wang and P. Zhang give a proof for $V_1 \neq 0$ (variable refraction index) in [WZ06] using uniform estimates given by Mourre method. We also mention the work of E. Fouassier who considered the case of a source which concentrates on two points (see [Fou06], $V_1 = 0$ in this case) and the case of a potential discontinuous along an affine hyperplane in [Fou07] (the source concentrates on 0 in this case). All this papers use *a priori* estimates of the solution in Besov spaces (we have already mentionned [PV99], see also [CJ06, WZ06, Wan07, CJK08] for further results about these estimates).

Here we are going to use the point of view of J.-F. Bony (see [Bon]). He considers the case of a source which concentrates on one or two points (with $V_1 \neq 0$) using a time-dependant method based on a BKW approximation of the propagator to prove that, microlocally, the solution of the Helmholtz equation is a finite sum of lagrangian distributions. In particular, abstract estimates of the solution are only used for the large times control, and this part of the solution has no contribution for the semiclassical measure, so the measure is actually constructed explicitely. Moreover, this method requires a geometrical assumption weaker than the Virial hypothesis used in the previous works.

In this paper we consider the case where not only the refraction index but also the absorption coefficient can be non-constant, and hence we have to work with a non-selfadjoint Schrödinger operator. But, as already mentionned, we know that the resolvent is well-defined for a spectral parameter z with $\mathrm{Im}\, z > 0$. For the selfadjoint semiclassical Schrödinger, we

[1] given with our notations.

need a non-trapping condition on classical trajectories of energy $E > 0$ to have uniform estimates of the resolvent and the limiting absorption principle around E (see [RT87, Wan87]). In the dissipative case, this assumption can be weakened as follows: any trajectory should either go to infinity or meet the region where $V_2 > 0$. This is the assumption we are going to use, and as as consequence, even if we can show that the outgoing solution u_h of (1.1) is microlocally zero in the incoming region, the contribution of large times in u_h does not vanishes when $h \to 0$ as is the case in [Bon], and in particular the solution can be an infinite sum of lagrangian distributions around some points of the phase space. However, the assumption that bounded trajectories should meet the region where there is absorption will make the series of amplitudes of these distributions convergent, which is the key argument in order to have a well-defined semiclassical measure in our case.

Concerning the source term, S_h is allowed to concentrate on any bounded submanifold of $\mathbb{R}^n$. We do not have problem like in [CPR02, WZ06] with decay assumptions, but this allows us to see what happens when the source concentrates on a non-flat submanifold. Note that we do not have phase factor in our source term (see below) so we are in the propagative regime described in [CPR02].

Let us now state the assumptions we are going to use in this work. We denote the free laplacian $-h^2\Delta$ by H_0^h and H_h is the dissipative Schrödinger operator on $L^2(\mathbb{R}^n)$ ($n \geqslant 1$):

$$H_h = -h^2\Delta + V_1(x) - ihV_2(x)$$

We also denote by $H_1^h = -h^2\Delta + V_1(x)$ the selfadjoint part of H_h. V_1, V_2 are smooth real functions on $\mathbb{R}^n$, V_2 is nonnegative and for $j \in \{1, 2\}$, $\alpha \in \mathbb{R}^n$:

$$|\partial^\alpha V_j(x)| \leqslant C_\alpha \langle x \rangle^{-\rho-|\alpha|} \tag{1.3}$$

for some $\rho > 0$. Here $\langle \cdot \rangle$ denotes the function $x \mapsto (1 + |x|^2)^{\frac{1}{2}}$. Let $p : (x, \xi) \mapsto \xi^2 + V_1(x)$ be the symbol on $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$ of the selfadjoint part H_1^h. The classical trajectories for this problem are the solutions $\phi^t(w) = (\overline{x}(t, w), \overline{\xi}(t, w))$ for $w \in \mathbb{R}^{2n}$ of the hamiltonian problem:

$$\begin{cases} \partial_t \overline{x}(t, w) = 2\overline{\xi}(t, w) \\ \partial_t \overline{\xi}(t, w) = -\nabla V_1(\overline{x}(t, w)) \\ \phi^0(w) = w \end{cases}$$

We recall from [Roy] that the exact hypothesis we need on an energy $E > 0$ to have the limiting absorption principle around E is the following: if we set

$$\mathcal{O} = \{x \in \mathbb{R}^n : V_2(x) > 0\}$$

then for all $w \in \mathbb{R}^{2n}$ such that $p(w) = E$ we have:

$$\left\{\phi^t(w), t \in \mathbb{R}\right\} \text{ is unbounded in } \mathbb{R}^{2n} \text{ or } \left\{\phi^t(w), t \in \mathbb{R}\right\} \cap \mathcal{O} \neq \emptyset \tag{1.4}$$

which means that any trapped trajectories should meet the set where there is absorption. For further use we also set, for $\gamma > 0$:

$$\mathcal{O}_\gamma = \{x \in \mathbb{R}^n : V_2(x) > \gamma\}$$

With this condition (which is actually necessary), for any $\alpha > \frac{1}{2}$ there exist $\varepsilon > 0$ and $c \geqslant 0$ such that:

$$\sup_{|\mathrm{Re}\, z - E| \leqslant \varepsilon, \mathrm{Im}\, z > 0} \left\| \langle x \rangle^{-\alpha} (H_h - z)^{-1} \langle x \rangle^{-\alpha} \right\| \leqslant \frac{c}{h}$$

and for all $\lambda \in [E - \varepsilon, E + \varepsilon]$ the limit:

$$(H_h - (E + i0))^{-1} := \lim_{\mu \to 0^+} (H_h - (E + i\mu))^{-1}$$

exists (and is a continuous function of λ) in the space of bounded operators from $L^{2,\alpha}(\mathbb{R}^n)$ to $L^{2,-\alpha}(\mathbb{R}^n)$, where $L^{2,\delta}(\mathbb{R}^n)$ stands for $L^2(\langle x\rangle^{2\delta}\,dx)$. Then for all $S_h \in L^{2,\alpha}(\mathbb{R}^n)$, $u_h = (H_h - (E+i0))^{-1}S_h \in L^{2,-\alpha}(\mathbb{R}^n)$ is the outgoing solution for (1.1).

About the classical hamiltonian problem, we use the following notations:

$$\Omega_b^{\pm}(J) = \left\{ w \in \mathbb{R}^{2n} : \{\overline{x}(t,w), \pm t \geqslant 0\} \text{ is bounded} \right\}$$
$$\Omega_\infty^{\pm}(J) = \left\{ w \in \mathbb{R}^{2n} : |\overline{x}(t,w)| \xrightarrow[t\to\pm\infty]{} +\infty \right\}$$

Note that $\Omega_\infty^{\pm}(J)$ is open if J is open and $\Omega_b^{\pm}(J)$ is closed if J is closed.

Let us now introduce the source term we consider. Given a (bounded) submanifold Γ_2 of dimension $d \in [\![0, n-1]\!]$ in $\mathbb{R}^n$ with the measure σ induced by the Lebesgue measure on $\mathbb{R}^n$, a smooth function A of compact support on Γ_2 and a Schwartz function $S \in \mathcal{S}(\mathbb{R}^n)$, we note for $x \in \mathbb{R}^n$:

$$S_h(x) = h^{\frac{1-n-d}{2}} \int_{z\in\Gamma} A(z) S\left(\frac{x-z}{h}\right) d\sigma(z) \tag{1.5}$$

We can choose Γ and Γ_1 open in Γ_2 such that $\Gamma_0 := \operatorname{supp} A \subset \Gamma$, $\overline{\Gamma} \subset \Gamma_1$ and $\overline{\Gamma_1} \subset \Gamma_2$ (if Γ_2 is compact we can have $\Gamma_0 = \Gamma = \Gamma_1 = \Gamma_2$).

As usual, for $z \in \Gamma_2$ and $\zeta \in T_z\Gamma_2$ small enough (where $T_z\Gamma_2$ is the tangent space to Γ_2 at z), we denote by $\exp_z(\zeta)$ the point $c_\zeta(1)$ where $t \mapsto c_\zeta(t)$ is the unique geodesic on Γ_2 with initial conditions $c_\zeta(0) = z$ and $c'_\zeta(0) = \zeta$ (see [GHL90, §2.86]). On Γ_2 we define the distance d_Γ as usual: for $x, y \in \Gamma_2$, $d_\Gamma(x,y)$ is the infimum of the length of all piecewise C^1 curves from x to y. For $z \in \Gamma_2$, there exists a neighborhood $\mathcal{U}$ of z in Γ_2 and $\varepsilon > 0$ such that for $x, y \in \mathcal{U}$ there is a unique geodesic c from x to y of length less than ε. And the length of c is $d_\Gamma(x,y)$ (see [GHL90, §2.C.3]).

We consider a family of energies $E_h \in \mathbb{C}$ for $h \in]0,1]$. We assume that $\operatorname{Im} E_h \geqslant 0$ and:

$$E_h = E_0 + hE_1 + \underset{h\to 0}{o}(h) \tag{1.6}$$

where $E_0 > 0$ satisfies (1.4) and:

$$\forall z \in \overline{\Gamma}, \quad V_1(z) < E_0 \tag{1.7}$$

We set $N\Gamma = \{(z,\xi) \in \Gamma \times \mathbb{R}^n : \xi \perp T_z\Gamma\}$,

$$N_E\Gamma = \left\{ (z,\xi) \in N\Gamma : |\xi| = \sqrt{E_0 - V_1(z)} \right\}$$

and:

$$\Lambda = \left\{ \phi^t(z,\xi); t > 0, (z,\xi) \in N_E\Gamma \right\}$$

We similarly define $N_E\Gamma_0$ and $N_E\Gamma_1$. For $(z,\xi) \in N_E\Gamma$ and $(Z,\Xi) \in T_{(z,\xi)}N_E\Gamma$ we have $Z \in T_z\Gamma$ and $\Xi \in \mathbb{R}^n$ decomposes as $\Xi = \Xi_T + \Xi_{/\!/} + \Xi_\perp$ with $\Xi_T \in T_z\Gamma$, $\Xi_{/\!/} \in \mathbb{R}\xi$ and $\Xi_\perp \in (T_z\Gamma \oplus \mathbb{R}\xi)^\perp$. Then $N_E\Gamma$ is endowed with the metric g defined by:

$$g_{(z,\xi)}\left((Z^1,\Xi^1),(Z^2,\Xi^2)\right) = \left\langle Z^1, Z^2\right\rangle_{\mathbb{R}^n} + \left\langle \Xi^1_\perp, \Xi^2_\perp\right\rangle_{\mathbb{R}^n}$$

for all $(Z^1,\Xi^1),(Z^2,\Xi^2) \in T_{(z,\xi)}N_E\Gamma$. This means that we do not take into account the part of Ξ colinear to ξ and $T_z\Gamma$, which is allowed since (Z,Ξ) never reduces to $(0, \Xi_T + \Xi_{/\!/})$ unless $(Z,\Xi) = (0,0)$. Indeed, if $Z = 0$ then $\Xi \in T_{(z,\xi)}(N_E\Gamma \cap N_z\Gamma)$ and hence $\Xi = \Xi_\perp$. Now we denote by $\tilde{\sigma}$ the canonical measure on $N_E\Gamma$ given by the metric g. This means that for any smooth map $\psi : \mathcal{U} \to \mathcal{V}$ (where $\mathcal{U}$ is an open set in $\mathbb{R}^{n-1}$ and $\mathcal{V}$ is an open set in $N_E\Gamma$) and any function f on $\mathcal{V}$ we have (see [GHL90, §3.H]):

$$\int_{\mathcal{V}} f(v)\,d\tilde{\sigma}(v) = \int_{\mathcal{U}} f(\psi(u)) \left(\det(g_{\psi(u)}(\partial_i\psi(u), \partial_j\psi(u)))_{1\leqslant i,j\leqslant n-1}\right)^{\frac{1}{2}} du$$

Finally we set:

$$\Phi_0 = \left\{(z,\xi) \in N_E\Gamma : \exists t > 0, \phi^t(z,\xi) \in N_E\Gamma\right\}$$

The last assumption we need is:

$$\tilde{\sigma}(\Phi_0) = 0 \tag{1.8}$$

In [Bon, section 4] is given an example of what can happen without an hypothesis of this kind. Note that when $\Gamma = \{0\}$, this assumption is weaker than the assumption $\nu_0(E_0 - V_1(x)) - x.\nabla V_1(x) \geqslant c_0 > 0$ for some $\nu_0 \in]0,2]$ which is used for instance in [Wan07]. This is no longer true in general (for instance we can take $V_1 = 0$, $E_0 = 1$ and any circle in $\mathbb{R}^2$ for Γ).

To study semiclassical measures of u_h, we choose the point of view of pseudo-differential operators. Let us recall that the Weyl quantization of an observable $a : \mathbb{R}^{2n} \to \mathbb{C}$ is the operator:

$$Op_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a\left(\frac{x+y}{2},\xi\right) u(y)\,dy\,d\xi$$

We also use the standard quantization:

$$Op_h(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi)u(y)\,dy\,d\xi$$

See [Rob87, Mar02, EZ] for more details about semiclassical pseudo-differential operators, [Gér91] for semiclassical measures. We are going to use the following classes of symbols. For $\delta \in \mathbb{R}$ we set:

$$\mathcal{S}_\delta = \left\{a \in C^\infty(\mathbb{R}^{2n}) : \forall \alpha,\beta \in \mathbb{N}^n, \exists c_{\alpha,\beta}, \forall (x,\xi) \in \mathbb{R}^{2n}, \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \leqslant c_{\alpha,\beta} \langle x\rangle^{\delta - |\alpha|}\right\}$$

while $\mathcal{S}_b$ is the set of $C^\infty(\mathbb{R}^{2n})$ functions whose derivatives up to any order are in $L^\infty(\mathbb{R}^{2n})$.

We can now state the main theorem of this paper:

Theorem 1.1. *There exists a Radon measure μ on $\mathbb{R}^{2n}$ such that for all $q \in C_0^\infty(\mathbb{R}^{2n})$:*

$$\langle Op_h^w(q)u_h, u_h\rangle \xrightarrow[h\to 0]{} \int_{\mathbb{R}^{2n}} q\,d\mu$$

Moreover μ is characterized by the following three properties:

(i) μ is supported on the hypersurface of energy E_0:

$$\operatorname{supp}\mu \subset p^{-1}(\{E_0\})$$

(ii) μ vanishes in the incoming region: let $\sigma \in]0,1[$, then there exists $R \geqslant 0$ such that for $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in the incoming region $\Gamma_-(R,-\sigma)$ (see definition in section 5.1) we have:

$$\int q\,d\mu = 0$$

(iii) μ satisfies the Liouville equation:

$$(H_p + 2\operatorname{Im} E_1 + 2V_2)\mu = \pi(2\pi)^{d-n} A(z)^2 |\xi|^{-1} \hat{S}(\xi)^2 \tilde{\sigma} \tag{1.9}$$

where $H_p = \{p,\cdot\} = 2\xi.\partial_x - \nabla V_1(x).\partial_\xi$ and $\tilde{\sigma}$ is extended by 0 on $\mathbb{R}^{2n} \setminus N_E\Gamma$. This means that for any $q \in C_0^\infty(\mathbb{R}^{2n})$ we have:

$$\int_{\mathbb{R}^{2n}} (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu = \pi(2\pi)^{d-n} \int_{N_E\Gamma} q(z,\xi)A(z)^2 |\xi|^{-1} \hat{S}(\xi)^2\,d\tilde{\sigma}(z,\xi)$$

We first remark that this theorem gives not only existence of a semiclassical measure but also uniqueness, since we do not need to extract a subsequence to have convergence of $\langle Op_h^w(q)u_h, u_h\rangle$ when $h \to 0$.

Moreover, we see that in the Liouville equation the absorption coefficient α of (1.2) is replaced by our full non-constant absorption coefficient $\operatorname{Im} E_1 + V_2$, as one could expect.

And finally we will prove that the three properties of the theorem implies that the measure μ is given, for $q \in C_0^\infty(\mathbb{R}^{2n})$, by:

$$\int_{\mathbb{R}^{2n}} q\, d\mu = \pi(2\pi)^{d-n} \int_{\mathbb{R}_+} \int_{N_E\Gamma} A(z)^2 \,|\xi|^{-1} \hat{S}(\xi)^2 q(\phi^t(z,\xi)) e^{-2t \operatorname{Im} E_1 - 2\int_0^t V_2(\overline{x}(s,z,\xi))\, ds}\, d\tilde{\sigma}(z,\xi)\, dt \tag{1.10}$$

To prove this theorem we write as in [Bon] the resolvent as the integral over positive times of the propagator, the main difference being the large times contribution. Let:

$$U_h(t) = e^{-\frac{it}{h}H_h}, \quad U_0^h(t) = e^{-\frac{it}{h}H_0^h}, \quad \text{and} \quad U_h^E(t) = e^{-\frac{it}{h}(H_h - E_h)}$$

Then:

$$u_h = (H_h - (E_h + i0))^{-1} S_h = \frac{i}{h}\int_0^{+\infty} U_h^E(t) S_h\, dt \tag{1.11}$$

and for $T \geqslant 0$ we set:

$$\begin{aligned} u_h^T &= (H_h - (E_h + i0))^{-1} S_h - (H_h - (E_h + i0))^{-1} U_h^E(T) S_h \\ &= \frac{i}{h}\int_0^T U_h^E(t) S_h\, dt \end{aligned} \tag{1.12}$$

Our purpose is to study the quantity:

$$\lim_{h\to 0} \lim_{T\to+\infty} \left\langle Op_h(q) u_h^T, u_h^T \right\rangle$$

which we cannot do directly. Around $w \in \mathbb{R}^{2n}$, troubles appear when proving that relevant parts of integral (1.11) are around times t for which we can find $(z,\xi) \in N_E\Gamma$ such that $\phi^t(z,\xi) = w$ (see proposition 4.1). Indeed, far from these times we can find t such that $\phi^t(N_E\Gamma)$ is close to w, giving contribution for the semiclassical measure in any neighborhood of w. Moreover, the Egorov theorem we use gives estimates uniform in h but not in time (see [BR02] for a discussion of this problem). The key of our proof is to check that even if the contribution of large times is not zero as for the non-trapping case, the damping term V_2 makes it so small that the semiclassical measure is also given by:

$$\lim_{T\to+\infty} \lim_{h\to 0} \left\langle Op_h(q) u_h^T, u_h^T \right\rangle$$

which is much easier to study. Indeed, this means that we study the semiclassical measure for the family (u_h^T). This can be done as for the non-trapping case since we do not have to worry about large times behavior. This gives a family of measures on $\mathbb{R}^{2n}$, and then we can take the limit $T \to +\infty$, since we no longer have problems with the parameter h. It only remains to check this gives the measure we are looking for.

We begin the proof by a few preliminary results: we show to what extent the damping term V_2 implies a decay of $U_h(t)$, we look at the classical trajectories around the submanifold Γ and give more details about the assumption on Φ_0. Finally we show that the solution u_h concentrates on the hypersurface of energy E_0. In section 3 we give an estimate of the solution near Γ, since we cannot give a precise description of u_h there. This part is close to section 3.3 of [Bon] but we give a complete proof in order to see how to deal with the general case $\dim \Gamma \geqslant 1$. In section 4 we study the finite times contribution and give the semiclassical measure for u_h^T, and then in section 5 we prove that taking the limit $T \to +\infty$ for this family of measures gives a semiclassical measure for the solution u_h. We also show that this limit is the solution of the Liouville equation (1.9) where V_2 naturally appears as a damping factor.

Finally in section 6 we give the proof of the estimate in the incoming region we use in section 5. Indeed if we no longer assume that all the classical trajectories of energy E_0 go to infinity, there still are some non-trapped trajectories. So we still need the estimate of the outgoing solution in the incoming region used in the non-trapping case. For the self-adjoint Schrödinger operator, this is proved in [RT89] but here we need to show that this remains true in our dissipative setting.

2 Some preliminary results

2.1 Damping effect of the absorption coefficient on the semigroup generated by H_h

We saw in [Roy] that assumption (1.4) is actually satisfied for any energy close enough to E_0, hence we can consider two closed intervals I and J such that $E_0 \in \mathring{I}$, $\overline{I} \subset \mathring{J}$ and any trapped trajectory of energy in J meets $\mathcal{O}$.

The main tool we need in this section is the dissipative version of Egorov theorem. We already stated this theorem in [Roy] but we give here a more precise version we are going to use in the proof of proposition 4.1.

Proposition 2.1. *Let $a \in \mathcal{S}_b$.*

(i) There exists a family of symbols $\alpha_j(t)$ for $j \in \mathbb{N}$ and $t \geqslant 0$ such that for any $N \in \mathbb{N}$ and $t \geqslant 0$ the symbol $A_N(t,h) = \sum_{j=0}^N h^j \alpha_j(t)$ satisfies:

$$U_h(t)^* Op_h^w(a) U_h(t) = Op_h^w(A_N(t,h)) + \underset{h\to 0}{O}(h^{N+1})$$

where the rest is bounded as an operator on $L^2(\mathbb{R}^n)$ uniformly in $t \in [0,T]$ for any $T \geqslant 0$.

(ii) $\alpha_0(t) = (a \circ \phi^t)\exp\left(-2\int_0^t V_2 \circ \phi^s\, ds\right)$ where for $(x,\xi) \in \mathbb{R}^{2n}$, $V_2(x,\xi)$ means $V_2(x)$.

(iii) If a vanishes on the open set $\mathcal{W} \subset \mathbb{R}^{2n}$ then for all $j \in \mathbb{N}$ the symbol $\alpha_j(t)$ vanishes on $\phi^{-t}(\mathcal{W})$.

Proof. In [Roy] we proved (i) for $N = 0$ and (ii). Moreover (iii) is a direct consequence of (ii) for $j = 0$. What remains can be proved as in the selfadjoint case (see [Rob87]) so we only recall the ideas. (i) is proved by induction. More precisely, we show that for any $N \in \mathbb{N}$:

$$\begin{aligned} U_h(t)^* Op_h^w(a) U_h(t) &= \sum_{j=0}^N h^j Op_h^w(\alpha_j(t)) \\ &+ h^{N+1}\int_{\tau_1=0}^t \int_{\tau_2=0}^{\tau_1} \dots \int_{\tau_{N+1}=0}^{\tau_N} U_h(\tau_{N+1})^* Op_h^w(b_N(\tau_1,\dots,\tau_{N+1},h)) U_h(\tau_{N+1})\, d\tau_{N+1}\dots d\tau_1 \end{aligned}$$

for some symbol b_N. The case $N+1$ is obtained by applying the case $N = 0$ to the principal symbol of b_N.

To prove (iii) we take the derivative of $U_h(t)^* Op_h^w(a) U_h(t)$ with report to t. This gives, for $j \in \mathbb{N}$:

$$\partial_t \alpha_j(t) = H_p(\alpha_j) - 2V_2\alpha_j(t) + \sum_{q=0}^{j-1} C_{j,q} D_{j,q}^* \alpha_q$$

where $C_{j,q}$ is a function with bounded derivatives and $D^*_{j,q}$ is a differential operator. Then if $\tilde{\alpha}_j(t) = (\alpha_j(t) \circ \phi^{-t}) \exp\left(2\int_0^t V_2 \circ \phi^{-s}\, ds\right)$ we have:

$$\partial_t \tilde{\alpha}_j(t) = \sum_{q=0}^{j-1} C_{j,q} D^*_{j,q}(\alpha_q(t) \circ \phi^{-t}) \exp\left(2\int_0^t V_2 \circ \phi^{-s}\, ds\right)$$

and it is easy to check by induction on $j \geqslant 1$:

$$\tilde{\alpha}_j(0) = 0, \quad \partial_t \tilde{\alpha}_j(t) = 0 \text{ on } \mathcal{W}, \quad \text{and hence } \alpha_j(t) = 0 \text{ on } \phi^{-t}(\mathcal{W})$$

□

Lemma 2.2. *Let K be a compact subset of $\Omega_b^+(J)$. There is $C \geqslant 0$ and $\delta > 0$ such that:*

$$\forall w \in K, \quad \exp\left(-\int_{s=0}^t V_2(\phi^s(w))\, ds\right) \leqslant C e^{-\delta t}$$

Proof. **1.** We first recall that if $w \in \Omega_b^+(J)$ then there exists $T \geqslant 0$ such that $\phi^T(w) \in \mathcal{O}$ (this is slightly stronger than assumption (1.4)). Indeed, the set $K_w = \overline{\{\phi^t(w), t \geqslant 0\}}$ is compact, so there is an increasing sequence $(t_m)_{m\in\mathbb{N}}$ with $t_m \to +\infty$ and $w_\infty \in K_w$ such that $\phi^{t_m}(w) \to w_\infty$. Since $\Omega_b^+(\{p(w)\})$ is closed, $w_\infty \in \Omega_b^+(\{p(w)\})$. Moreover, for $M \in \mathbb{N}$ and $m \geqslant M$ we have $\phi^{-t_M}(\phi^{t_m}(w)) \in K_w$ and hence $\phi^{-t_M}(w_\infty) \in K_w$, which proves that $w_\infty \in \Omega_b^-(\mathbb{R})$. By assymption (1.4), there is $T \in \mathbb{R}$ such that $\phi^T(w_\infty) \in \mathcal{O}$. Hence $\phi^{T+t_m}(w)$ lies in $\mathcal{O}$ for large m. Since $T + t_m \geqslant 0$ when m is large enough, the claim is proved.

2. We set:

$$\tilde{K} = \overline{\{\phi^t(w), t \geqslant 0, w \in K\}}$$

By definition of K, $\tilde{K}$ is compact in $\mathbb{R}^{2n}$. Let $w \in \tilde{K}$. There are $T_w \geqslant 0$ and $\gamma_w > 0$ such that $\phi^{T_w}(w) \in \mathcal{O}_{2\gamma_w}$, so we can find $\tau_w > 0$ and a neighborhood $\mathcal{V}_w$ of w in $\mathbb{R}^{2n}$ such that for all $v \in \mathcal{V}_w$ and $t \in [T_w - \tau_w, T_w]$ we have: $\phi^t(v) \in \mathcal{O}_{\gamma_w}$. As $\tilde{K}$ is compact we can find $w_1, \dots, w_k$ such that $K \subset \cup_{i=1}^k \mathcal{V}_{w_i}$. Then we take $T = \max\{T_{w_i}, 1 \leqslant i \leqslant k\}$, $\tau = \min\{\tau_i, 1 \leqslant i \leqslant k\}$ and $\gamma = \min\{\gamma_{w_i}, 1 \leqslant i \leqslant k\}$. For all $w \in K$ and $t \geqslant 0$, $\phi^t(w)$ is in $\tilde{K}$ and hence in $[t, t+T]$ there is a subinterval $I_{w,t}$ of length at least τ such that $\phi^s(w) \in \mathcal{O}_\gamma$ for $s \in I_{w,t}$. Thus:

$$\exp\left(\int_{s=t}^{t+T} V_2(\phi^s(w))\, ds\right) \leqslant e^{-\tau\gamma}$$

We apply this for $t_n = nT$ with $n \leqslant t/T$ and this gives:

$$\exp\left(\int_0^t V_2(\phi^s(w))\, ds\right) \leqslant e^{-\frac{t-T}{T}\tau\gamma} \leqslant e^{\tau\gamma} e^{-t\frac{\tau\gamma}{T}}$$

so the result follows with $C = e^{\tau\gamma}$ et $\delta = \frac{\tau\gamma}{T}$. □

Proposition 2.3. *Let $q, q' \in C_0^\infty(\mathbb{R}^{2n})$ supported in $p^{-1}(J)$ and $\varepsilon > 0$. Then there exists $T_0 \geqslant 0$ such that for all $T \geqslant T_0$ we can find $h_T > 0$ which satisfies:*

$$\forall h \in]0, h_T], \quad \|Op_h^w(q) U_h(T) Op_h^w(q')\| \leqslant \varepsilon$$

Proof. We set $K = \operatorname{supp} q' \cap \Omega_b^+(\mathbb{R})$. As K is a compact subset of $\Omega_b^+(J)$, lemma 2.2 shows that there is $T_0 \geqslant 0$ such that:

$$\sup_{w\in K} \|q\|_\infty \|q'\|_\infty \exp\left(-\int_{s=0}^T (V_2 \circ \phi^s)(w)\, ds\right) \leqslant \frac{\varepsilon}{4}$$

As the left-hand side is a continuous function of w, we can find a neighborhood $\mathcal{V}$ of K in $\mathbb{R}^{2n}$ such that this holds for $w \in \mathcal{V}$ after having replaced $\varepsilon/4$ by $\varepsilon/2$. Let now $K_\infty = \operatorname{supp} q' \setminus \mathcal{V}$. K_∞ is a compact subset of Ω_+^∞. Therefore, if T_0 is large enough, we can assume that for $T \geqslant T_0$ and $w \in K_\infty$ we have $\phi^T(w) \notin \operatorname{supp} q$. Hence by Egorov theorem (see also remark 4.4 in [Roy]), for any $T \geqslant T_0$ we have:

$$\begin{aligned}
\|Op_h^w(q)U_h(T)Op_h^w(q')\| &= \left\|U_1^h(-T)Op_h^w(q)U_h(T)Op_h^w(q')\right\| \\
&= \left\|Op_h^w\left((q\circ\phi^T)e^{-\int_{s=0}^T V_2\circ\phi^s\,ds}\right)Op_h^w(q')\right\| + \underset{h\to 0}{O}(h) \\
&\leqslant \sup_{w\in\mathbb{R}^{2n}}\left|q'(w)(q(\phi^T(w)))e^{-\int_{s=0}^T V_2(\phi^s(w))\,ds}\right| + C(T)\sqrt{h} \\
&\leqslant \frac{\varepsilon}{2} + C(T)\sqrt{h}
\end{aligned} \tag{2.1}$$

and hence for any fixed $T \geqslant T_0$ we can find $h_T > 0$ small enough to conclude. $\square$

2.2 Classical trajectories around Γ

In this section we assume that assumptions (1.3), (1.4) and (1.7) are satisfied.

Proposition 2.4. *There exists $\tau_0 > 0$ such that:*

$$\mathcal{T} : \begin{cases}]0, 3\tau_0] \times N_E\Gamma_1 & \to \mathbb{R}^n \\ (t, w) & \mapsto \overline{x}(t, w) \end{cases} \tag{2.2}$$

is one-to-one and $\operatorname{Ran}(\mathcal{T}) \cup \Gamma_1$ *is a neighborhood of* Γ *in* $\mathbb{R}^n$*. Furthermore:*

(i) We can choose τ_0 to have:

$$\forall t \in]0, 3\tau_0], \forall w \in N_E\Gamma_1, \quad 2\gamma_m t \leqslant d(\overline{x}(t, w), \Gamma_2) \leqslant 2\gamma_M t \tag{2.3}$$

for some $\gamma_M \geqslant \gamma_m > 0$.

(ii) If f is a continuous function with support in $\mathcal{T}(]0, 3\tau_0[\times N_E\Gamma)$ then:

$$\int_{x\in\mathbb{R}^n} f(x)\,dx = 2^{n-d}\int_0^{3\tau_0}\int_{N_E\Gamma} f(\overline{x}(t, z, \xi))t^{n-d-1}\,|\xi|\left(1 + \underset{t\to 0}{O}(t)\right)d\tilde{\sigma}(z, \xi)\,dt \tag{2.4}$$

For $0 \leqslant r_1 \leqslant r_2 \leqslant 3\tau_0$ we set:

$$\tilde{\Gamma}(r_2) = \mathcal{T}([0, r_2] \times N_E\Gamma) \quad \text{and} \quad \tilde{\Gamma}(r_1, r_2) = \mathcal{T}(]r_1, r_2] \times N_E\Gamma)$$

When $x \in \tilde{\Gamma}(0, 3\tau_0)$ we write $(t_x, z_x, \xi_x) = \mathcal{T}^{-1}(x)$.

Proof. For $\tau > 0$, let :

$$N(\tau) = \left\{(z, \xi) \in N\Gamma_1 : |\xi| \leqslant \tau\sqrt{E_0 - V_1(z)}\right\}$$

We consider the function $\tilde{\mathcal{T}}$ from $N(1)$ to $\mathbb{R}^n$ defined by:

$$\tilde{\mathcal{T}}(z, \xi) = \begin{cases} \overline{x}\left(\frac{|\xi|}{\sqrt{E_0 - V_1(z)}}, z, \frac{\xi\sqrt{E_0 - V_1(z)}}{|\xi|}\right) & \text{if } \xi \neq 0 \\ z & \text{if } \xi = 0 \end{cases}$$

We have:

$$\tilde{\mathcal{T}}(z, \xi) = z + 2\xi + o(|\xi|)$$

Hence for $\tau_0 > 0$ small enough, $\tilde{\mathcal{T}}$ is a diffeomorphism from $N(3\tau_0)$ to a tubular neighborhood of Γ_1 (we can follow the proof for the function $(x, \xi) \mapsto z + 2\xi$, see for instance theorem

2.7.12 in [BG87]). In particular $\tilde{\mathcal{T}}$ and hence $\mathcal{T} : (t,z,\xi) \mapsto \tilde{\mathcal{T}}(z,t\xi)$ are one-to-one and $\operatorname{Ran}\mathcal{T} \cup \Gamma_1 = \operatorname{Ran}\tilde{\mathcal{T}}\Big|_{N(3\tau_0)} \cup \Gamma_1$ is a neighborhood of Γ_0.

(i) We have:

$$\overline{x}(t,z,\xi) - z = \int_0^t 2\overline{\xi}(s,z,\xi)\,ds = 2t\xi - 2\int_0^t \int_0^s \nabla V_1(u,z,\xi)\,du\,ds$$

Hence, if $M = \sup_{x\in\mathbb{R}^n} |\nabla V_1(x)|$ this gives:

$$|\overline{x}(t,z,\xi) - z - 2t\xi| \leqslant 2t^2 M$$

Denote $\xi_{\min} = \min\{|\xi|, \xi \in N_E\Gamma_1\} > 0$ and $\xi_{\max} = \max\{|\xi|, \xi \in N_E\Gamma_1\}$. We recall from [BG87] that for $(z,\xi) \in N_E\Gamma_1$ and t small enough we have $d(z + t\xi, \Gamma_2) = t|\xi|$. Then for τ_0 small enough we have $2\tau_0 M \leqslant \xi_{\min}$ so:

$$d(\overline{x}(t,z,\xi),\Gamma_2) \geqslant d(z+2t\xi,\Gamma_2) - |\overline{x}(t,z,\xi) - z - 2t\xi| \geqslant 2t|\xi| - t\xi_{\min} \geqslant t\xi_{\min}$$

and:

$$d(\overline{x}(t,z,\xi),\Gamma_2) \leqslant d(z+2t\xi,\Gamma_2) + |\overline{x}(t,z,\xi) - z - 2t\xi| \leqslant 2t|\xi| + t\xi_{\min} \leqslant t(2\xi_{\max} + \xi_{\min})$$

(ii) Let $(t,z,\xi) \in]0,3\tau_0[\times N_E\Gamma$. For $(T_1,Z_1,\Xi_1),(T_2,Z_2,\Xi_2) \in T_{(t,z,\xi)}(]0,3\tau_0[\times N_E\Gamma)$ we set:

$$\tilde{g}_{(t,z,\xi)}((T_1,Z_1,\Xi_1),(T_2,Z_2,\Xi_2)) = T_1T_2 + g_{(z,\xi)}((Z_1,\Xi_1),(Z_2,\Xi_2))$$

We first look for good orthonormal bases of $T_{(t,z,\xi)}(]0,3\tau_0[\times N_E\Gamma)$ (for the metric $\tilde{g}$) and $\mathbb{R}^n$ (for the usual metric) to compute the jacobian of $\mathcal{T}$. $N_E\Gamma \cap (\{z\}\times\mathbb{R}^n)$ is a submanifold of dimension $n-d-1$ in $N_E\Gamma$, so we can consider an orthonormal basis $((0,\Xi_j))_{d+2\leqslant j\leqslant n}$ of its tangent space at (z,ξ). We now choose an orthonormal basis $(Z_j)_{2\leqslant j\leqslant d+1}$ of $T_z\Gamma$. We can find $\Xi_2,\dots,\Xi_{d+1} \in \mathbb{R}^n$ such that $(Z_j,\Xi_j) \in T_{(z,\xi)}N_E\Gamma$ for $j \in [\![2,d+1]\!]$ and since linear combinations of $(0,\Xi_{d+2}),\dots,(0,\Xi_n)$ can be added, we may assume that $\Xi_j \in T_z\Gamma \oplus \mathbb{R}\xi$ for all $j \in [\![2,d+1]\!]$. These $n-1$ vectors form an orthonormal family of $T_{(z,\xi)}N_E\Gamma$ to which we add the canonical unit vector of $\mathbb{R}$ for the time component. This gives an orthonormal basis $\mathcal{B}_{(t,z,\xi)}$ of $T_{(t,z,\xi)}(]0,3\tau_0[\times N_E\Gamma)$. In $\mathbb{R}^n$ we consider the orthonormal basis:

$$\tilde{\mathcal{B}}_{\mathcal{T}(t,z,\xi)} = (\xi/|\xi|, Z_{n-d},\dots,Z_{n-1},\Xi_1,\dots,\Xi_{n-d-1})$$

Since $\mathcal{T}(t,z,\xi) = z + 2t\xi + O(t^2)$, the jacobian matrix of $\mathcal{T}$ in these two bases is:

$$\operatorname{Mat}_{\mathcal{B}_{(t,z,\xi)}\to\tilde{\mathcal{B}}_{\mathcal{T}(t,z,\xi)}} D_{(t,z,\xi)}\mathcal{T} = \begin{pmatrix} 2|\xi| & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 2tI_{n-d-1} \end{pmatrix} \left(1 + \underset{t\to 0}{O}(t)\right)$$

On the other hand, since basis $\mathcal{B}_{(t,z,\xi)}$ and $\tilde{\mathcal{B}}_{\mathcal{T}(t,z,\xi)}$ are orthonormal, we have, for $x \in \tilde{\Gamma}(0,3\tau_0)$:

$$\left(\det(\tilde{g}_{\mathcal{T}^{-1}(x)}(\partial_i\mathcal{T}^{-1}(x),\partial_j\mathcal{T}^{-1}(x)))_{1\leqslant i,j\leqslant n}\right)^{\frac{1}{2}} = \left|\det\operatorname{Mat}_{\tilde{\mathcal{B}}_x\to\mathcal{B}_{\mathcal{T}^{-1}(x)}} D_x\mathcal{T}^{-1}\right|$$

Thus, using the definition of the measure $dt\,d\tilde{\sigma}$ on $]0,3\tau_0[\times N_E\Gamma$ and the fact that $\mathcal{T}^{-1} : \tilde{\Gamma}(0,3\tau_0) \to]0,3\tau_0[\times N_E\Gamma$ can be seen as a map for the manifold $]0,3\tau_0[\times N_E\Gamma$, we obtain:

$$\begin{aligned}
&\int_{x\in\mathbb{R}^n} f(x)\,dx \\
&\quad= \int_{x\in\mathbb{R}^n} (f\circ\mathcal{T})(\mathcal{T}^{-1}x)\left|\det\operatorname{Mat}_{\tilde{\mathcal{B}}_x\to\mathcal{B}_{\mathcal{T}^{-1}(x)}} D_x\mathcal{T}^{-1}\right|\left|\det\operatorname{Mat}_{\mathcal{B}_{\mathcal{T}^{-1}(x)}\to\tilde{\mathcal{B}}_x} D_{\mathcal{T}^{-1}(x)}\mathcal{T}\right| dx \\
&\quad= \int_{t=0}^{3\tau_0}\int_{(z,\xi)\in N_E\Gamma} (f\circ\mathcal{T})(t,z,\xi)\left|\det\operatorname{Mat}_{\mathcal{B}_{(t,z,\xi)}\to\tilde{\mathcal{B}}_{\mathcal{T}(t,z,\xi)}} D_{(t,z,\xi)}\mathcal{T}\right| d\tilde{\sigma}(z,\xi)\,dt \\
&\quad= 2^{n-d}\int_0^{3\tau_0}\int_{N_E\Gamma} f(\mathcal{T}(t,z,\xi))t^{n-d-1}|\xi|\left(1+\underset{t\to 0}{O}(t)\right) d\tilde{\sigma}(z,\xi)\,dt
\end{aligned}$$

□

Corollary 2.5. *Let* $(t,z,\xi) \neq (s,\zeta,\eta) \in \mathbb{R}_+^* \times N_E\Gamma$ *such that* $\phi^t(z,\xi) = \phi^s(\zeta,\eta)$*. Then* $|t-s| \geqslant 3\tau_0$ *where* τ_0 *is given by proposition 2.4.*

Let $w \in \mathbb{R}^{2n}$ and denote:

$$((t_{w,k}, z_{w,k}, \xi_{w,k}))_{1\leqslant k \leqslant K_w} = \left\{(t,z,\xi) \in \mathbb{R}_+^* \times N_E\Gamma : \phi^t(z,\xi) = w\right\}$$

with $t_{w,1} < t_{w,2} < \dots$ and $K_w \in \mathbb{N} \cup \{\infty\}$ ($[\![1, K_w]\!]$ is to be understood as $\mathbb{N}^*$ if $K_w = \infty$ and $K_w = 0$ if $w \notin \Lambda$). We also define $K_w^T = \sup\{k \in [\![1,K_w]\!] : t_{w,k} \leqslant T\} \in \mathbb{N}$. For $w \in \mathbb{R}^{2n}$ and $k \in [\![1,K_w]\!]$ we write:

$$\Lambda_{w,k} = \left\{\phi^t(z,\xi), |t - t_{w,k}| < \tau_0, |(z,\xi) - (z_k,\xi_k)| < \tau_0\right\}$$

and if $w \in N_E\Gamma$:

$$\Lambda_{w,0} = \left\{\phi^t(z,\xi), |t| < \tau_0, |(z,\xi) - w| < \tau_0\right\}$$

Proposition 2.6. *Let* $w = (x,\xi) \in \mathbb{R}^{2n}$ *and* $j,k \in [\![1,K_w]\!]$ *(*$[\![0,K_w]\!]$ *if* $w \in N_E\Gamma$*). Then*

(i) $\Lambda_{w,j} \cap \Lambda_{w,k}$ *is of measure zero in* $\Lambda_{w,j}$ *is and only if it is of measure zero in* $\Lambda_{w,k}$*.*

(ii) Assumption (1.8) *is equivalent to:*

$$\forall w \in \mathbb{R}^{2n}, \forall j,k \in [\![1,K_w]\!] (\textit{or } [\![0,K_w]\!]), \quad \Lambda_j \cap \Lambda_k \textit{ is of measure 0 in } \Lambda_j \tag{2.5}$$

This proposition is proved in section 6 of [Bon].

2.3 Localization around E_0-energy hypersurface

Proposition 2.7. *For any* $\delta \in \mathbb{R}$ *we have:*

$$\|S_h\|_{L^{2,\delta}(\mathbb{R}^n)} = \underset{h\to 0}{O}\left(\sqrt{h}\right) \tag{2.6}$$

Proof. **1.** There exists $C \geqslant 0$ such that for all $x \in \mathbb{R}^n$ and $r > 0$, the measure of $B(x,r) \cap \Gamma$ in Γ is less than Cr^d. Otherwise for all $m \in \mathbb{N}$ we can find $x_m \in \mathbb{R}^n$ and $r_m > 0$ such that the measure of the ball $B(x_m,r_m) \cap \Gamma$ in Γ is greater than mr_m^d. As Γ is of finite measure, r_m necessarily goes to 0 as $m \to +\infty$. On the other hand x_m has to stay close to Γ, hence in a compact subset of $\mathbb{R}^n$, so taking a subsequence we can assume that $x_m \to x_\infty \in \Gamma$. But the part of Γ close to x_∞ is diffeomorphic to a subset of $\mathbb{R}^d \subset \mathbb{R}^n$, hence the measure of $B(x_\infty, r) \cap \Gamma$ in Γ is less than Cr^d for some $C \geqslant 0$.

2. Let $x \in \mathbb{R}^n$. We have:

$$\begin{aligned} S_h(x)^2 &= h^{1-n-d} \left(\sum_{m\in\mathbb{N}} \int_{mh \leqslant |x-z| < (m+1)h} A(z) S\left(\frac{x-z}{h}\right) d\sigma(z) \right)^2 \\ &\leqslant c\, h^{1-n-d} \sum_{m\in\mathbb{N}} m^2 \left(\int_{mh \leqslant |x-z| < (m+1)h} A(z) S\left(\frac{x-z}{h}\right) d\sigma(z) \right)^2 \\ &\leqslant c\, h^{1-n} \sum_{m\in\mathbb{N}} m^{2+d} \int_{mh \leqslant |x-z| < (m+1)h} S\left(\frac{x-z}{h}\right)^2 d\sigma(z) \end{aligned}$$

and hence:

$$\begin{aligned} \|S_h\|^2_{L^{2,\delta}(\mathbb{R}^n)} &\leqslant c\, h^{1-n} \int_{x\in\mathbb{R}^n} \sum_{m\in\mathbb{N}} m^{2+d} \int_{mh \leqslant |x-z| < (m+1)h} \langle x \rangle^{2\delta} S\left(\frac{x-z}{h}\right)^2 d\sigma(z)\, dx \\ &\leqslant c\, h \sum_{m\in\mathbb{N}} m^{2+d} \int_{z\in\Gamma} \int_{m \leqslant |y| < (m+1)} \langle z + hy \rangle^{2\delta} S(y)^2 \, dy\, d\sigma(z) \\ &\leqslant c\, h \sum_{m\in\mathbb{N}} m^{2+d} \int_{z\in\Gamma} \int_{m \leqslant |y| < (m+1)} \langle y \rangle^{2\delta} S(y)^2 \, dy\, d\sigma(z) \end{aligned}$$

for $h \in]0,1]$, since Γ is bounded. As S decays faster than $\langle y \rangle^{-\frac{n+2\delta+4+d}{2}}$ we have:

$$\|S_h\|^2_{L^{2,\delta}(\mathbb{R}^n)} \leqslant c\,h \sum_{m\in\mathbb{N}} m^2 \langle m \rangle^{-4-d} \leqslant c\,h$$

□

Since $(H_h - (E_h + i0))^{-1} = O(h^{-1})$ as an operator from $L^{2,\alpha}(\mathbb{R}^n)$ to $L^{2,-\alpha}(\mathbb{R}^n)$ we get:

Corollary 2.8. *$u_h = \underset{h\to 0}{O}(h^{-\frac{1}{2}})$ in $L^{2,-\alpha}(\mathbb{R}^n)$. The same applies to u_h^T for all $T \geqslant 0$.*

Proposition 2.9. *S_h is microlocalized in $N\Gamma_0$.*

Proof. Let $q \in C_0^\infty(\mathbb{R}^{2n})$ supported outside $N\Gamma_0$. We have:

$$\begin{aligned} Op_h^w(q)S_h(x) &= \frac{1}{(2\pi h)^n}\int_\Gamma\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} q(x,\xi)A(z)S\left(\frac{y-z}{h}\right)dy\,d\xi\,d\sigma(z) \\ &= \frac{1}{(2\pi)^n}\int_\Gamma\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-z,\xi\rangle}e^{-i\langle v,\xi\rangle} q(x,\xi)A(z)S(v)\,dv\,d\xi\,d\sigma(z) \end{aligned}$$

If $\partial_z \langle x-z,\xi\rangle = 0$ and $\partial_\xi \langle x-z,\xi\rangle = 0$ then $x = z$ and $\xi \in N_z\Gamma$ so $A(z)q(x,\xi) = 0$. According to the non-stationnary phase theorem, we have $Op_h^w(q)S_h = O(h^\infty)$ in $L^2(\mathbb{R}^n)$. □

Proposition 2.10. *(i) Let $g \in \mathcal{S}_b$ equal to 1 in a neighborhood of $p^{-1}(\{E_0\})$. We have:*

$$\left\|Op_h^w(1-g)(H_h - (E_h + i0))^{-1}\right\|_{L^{2,\alpha}(\mathbb{R}^n)\to L^{2-\alpha}(\mathbb{R}^n)} = \underset{h\to 0}{O}(1) \tag{2.7}$$

(ii) Let $f \in \mathcal{S}_b$ equal to 1 in a neighborhood of $\overline{N_E\Gamma_0}$, then in $L^{2,-\alpha}(\mathbb{R}^n)$:

$$u_h = (H_h - (E_h + i0))^{-1}Op_h(f)S_h + \underset{h\to 0}{O}(\sqrt{h}) \tag{2.8}$$

(iii) Moreover there exists $\tilde{g} \in C_0^\infty(\mathbb{R})$ equal to 1 in a neighborhood of E_0 such that in $L^{2,-\alpha}(\mathbb{R}^n)$:

$$(H_h-(E_h+i0))^{-1}Op_h(1-f)S_h = (1-\tilde{g})(H_1^h)(H_h-(E_h+i0))^{-1}Op_h(1-f)S_h + \underset{h\to 0}{O}\left(h^{\frac{3}{2}}\right) \tag{2.9}$$

Similar results hold for u_h^T, $T \geqslant 0$.

Proof. (i) For $\operatorname{Im} z > 0$ we have:

$$Op_h(1-g)(H_h - z)^{-1} = Op_h(1-g)(H_1^h - z)^{-1}(1 + hV_2(H_h - z)^{-1})$$

According to [HR83] we have:

$$(H_1^h - z)^{-1} = Op_h^w\left((p(x,\xi) - z)^{-1}\right) + \underset{h\to 0}{O}(h)$$

Since $(p(x,\xi) - z)^{-1}$ is bounded on $\operatorname{supp}(1-g)$ uniformly for z close to E_0, $\operatorname{Im} z > 0$, the operator $Op_h^w(1-g)(H_1^h - z)^{-1}$ is uniformly bounded in $h > 0$ and z close to E_0, $\operatorname{Im} z > 0$. Moreover $(1 + hV_2(H_h - z)^{-1})$ is uniformly bounded as an operator from $L^{2,\alpha}(\mathbb{R}^n)$ to $L^{2,-\alpha}(\mathbb{R}^n)$ so:

$$\left\|Op_h^w(1-g)(H_h - z)^{-1}\right\|_{L^{2,\alpha}(\mathbb{R}^n)\to L^{2-\alpha}(\mathbb{R}^n)} = \underset{h\to 0}{O}(1)$$

uniformly in z. Taking the limit $z \to E_h + i0$ gives (2.7).

(ii) Let $\mathcal{U}$ be a neighborhood of $\overline{N_E\Gamma_0}$ in $\mathbb{R}^{2n}$ such that $f = 1$ on $\mathcal{U}$. We can find $\varepsilon > 0$ such that $p^{-1}([E_0 - 2\varepsilon, E_0 + 2\varepsilon]) \setminus \mathcal{U}$ does not intersect $\overline{N\Gamma_0}$. Let $\chi \in C_0^\infty(\mathbb{R})$ supported in

$]E_0-2\varepsilon, E_0+2\varepsilon[$ and equal to 1 on $]E_0-\varepsilon, E_0+\varepsilon[$. Since modulo $O(h^\infty)$ the operator $\chi(H_1)$ is a pseudo-differential operator with symbol supported in $\mathrm{supp}(\chi \circ p)$ and S_h is microlocalized on $N\Gamma_0$ we have in $L^{2,\alpha}(\mathbb{R}^n)$:

$$(H_h - (E_h + i0))^{-1} Op_h(1-f)\chi(H_1)S_h = \underset{h\to 0}{O}(h^\infty)$$

On the other hand, as we proved (2.7) we see that:

$$(H_h - (E_h + i0))^{-1}(1-\chi)(H_1^h) = \underset{h\to 0}{O}(1)$$

so (2.8) follows since $Op_h^w(1-f)S_h = O(\sqrt{h})$.

(iii) Let us refine this last estimate. Let $\tilde{g} \in C_0^\infty(\mathbb{R})$ supported in $[E_0-\varepsilon, E_0+\varepsilon]$ and equal to 1 in a neighborhood of E_0. Since $(1-\chi)\tilde{g} = 0$, we have:

$$\begin{aligned}
&\tilde{g}(H_1^h)(H_h - z)^{-1}(1-\chi)(H_1^h) \\
&\quad = \tilde{g}(H_1^h)(H_h - z)^{-1}(1-\chi)(H_1^h)(1-\tilde{g})(H_1^h) \\
&\quad = \tilde{g}(H_1^h)(1 + h(H_h - z)^{-1}V_2)(H_1^h - z)^{-1}(1-\chi)(H_1^h)(1-\tilde{g})(H_1^h) \\
&\quad = h\tilde{g}(H_1^h)\,(H_h - z)^{-1}V_2\,(1-\chi)(H_1^h)\,(1-\tilde{g})(H_1^h)\,(H_1^h - z)^{-1}
\end{aligned}$$

It only remains to see that the operators $(H_h-z)^{-1}V_2(1-\chi)(H_1^h)$ and $(1-\tilde{g})(H_1^h)(H_1^h-z)^{-1}$ are bounded uniformly in $h \in]0,1]$ and z close to E_0 with $\mathrm{Im}\, z > 0$. □

As a first consequence of this proposition we see that the solution u_h consentrates on $p^{-1}(\{E_0\})$:

Corollary 2.11. *If $q \in C_0^\infty(\mathbb{R}^n)$ has support outside $p^{-1}(\{E_0\})$ then:*

$$\langle Op_h^w(q)u_h, u_h\rangle \xrightarrow[h\to 0]{} 0$$

Proof. Let $\tilde{q} \in C_0^\infty(\mathbb{R}^{2n})$ supported outside $p^{-1}(\{E_0\})$ and equal to 1 on $\mathrm{supp}\, q$. We have:

$$\langle Op_h^w(q)u_h, u_h\rangle = \langle Op_h^w(q)u_h, Op_h^w(\tilde{q})u_h\rangle + \underset{h\to 0}{O}(h^\infty) = \underset{h\to 0}{O}(h)$$

□

3 Around Γ

3.1 WKB method

According to proposition IV.14 in [Rob87] or lemma 10.10 in [EZ] applied with the symbol $p_E : (x,\xi) \mapsto \xi^2 + V_1(x) - E_0$ we know that if τ_0 is small enough, then there exists a function $\varphi \in C^\infty([-3\tau_0, 3\tau_0] \times \mathbb{R}^{2n})$ such that:

$$\begin{cases} \partial_t\varphi(t,x,\xi) + |\partial_x\varphi(t,x,\xi)|^2 + V_1(x) - E_0 = 0 \\ \varphi(0,x,\xi) = \langle x, \xi\rangle \end{cases} \tag{3.1}$$

Moreover φ is unique and:

$$\begin{aligned}
\varphi(t,x,\xi) &= \langle \overline{y}(t,x,\xi), \xi\rangle + 2\int_0^t \tilde{\xi}(s,t,x,\xi)^2\, ds - tp_E(x,\xi) \\
&= \langle x,\xi\rangle - 2\int_0^t \left\langle \tilde{\xi}(s,t,x,\xi), \xi\right\rangle\, ds + 2\int_0^t \tilde{\xi}(s,t,x,\xi)^2\, ds - tp_E(x,\xi) \\
&= \langle x,\xi\rangle - tp_E(x,\xi) + t^2 r(t,x,\xi)
\end{aligned} \tag{3.2}$$

where $\overline{y}(t,x,\xi)$ is the unique point in $\mathbb{R}^n$ such that $\overline{x}(t,\overline{y}(t,x,\xi),\xi) = x$ (note that $\overline{y}(t,x,\xi)$ is well-defined for t small enough, see [Rob87]) and:

$$r(t,x,\xi) = \frac{2}{t^2}\int_{s=0}^{t}\int_{\tau=s}^{t}\left\langle \tilde{\xi}(s,t,x,\xi), \nabla V_1(\tilde{x}(\tau,t,x,\xi))\right\rangle d\tau\, ds = \langle \xi, \nabla V_1(x)\rangle + \underset{t\to 0}{O}(t)$$

Proposition 3.1. *Let $f \in C_0^\infty(\mathbb{R}^{2n},\mathbb{R})$. We can find a function $a(h) \in C_0^\infty([0,3\tau_0]\times\mathbb{R}^{2n})$ such that:*

$$a(0,x,\xi,h) = f(x,\xi) \tag{3.3}$$

and:

$$\sup_{t\in[0,3\tau_0]}\left\| a(t,x,\xi,h)e^{\frac{i}{h}\varphi(t,x,\xi)} - e^{-\frac{it}{h}(H_h - E_h)}\left(f(x,\xi)e^{\frac{i}{h}\langle x,\xi\rangle}\right)\right\|_{L^2(\mathbb{R}^{2n})} \xrightarrow[h\to 0]{} 0 \tag{3.4}$$

Proof. We define:

$$\eta(s,t,x,\xi) = \exp\left(\int_s^t (iE_1 - V_2(\tilde{x}(\tau,t,x,\xi) - \Delta_x\varphi(\tau,\tilde{x}(\tau,t,x,\xi),\xi))\, d\tau\right)$$

Then:

$$a_0(t,x,\xi) = f(\overline{y}(t,x,\xi),\xi)\eta(0,t,x,\xi)$$

and:

$$a_1(t,y,\xi) = i\int_0^t \Delta_x a_0(s,\tilde{x}(s,t,x,\xi),\xi)\eta(s,t,x,\xi)\, ds$$

where for $0 \leqslant s \leqslant t \leqslant \tau_0$ we have set $\tilde{x}(s,t,x,\xi) = \overline{x}(s,\overline{y}(t,x,\xi),\xi)$. Then we set $a(h) = a_0 + ha_1$. Initial condition (3.3) is true and we can check that:

$$(\partial_t + 2\partial_x\varphi.\partial_x + \Delta_x\varphi + V_2 - iE_1)\, a_0(t,x,\xi) = 0$$

and:

$$(\partial_t + 2\partial_x\varphi.\partial_x + \Delta_x\varphi + V_2 - iE_1)\, a_1(t,x,\xi) = i\Delta_x a_0(t,x,\xi)$$

which, with (3.1), give (3.4). Note that the function $a(h)$ is of compact support and the absorption coefficient V_2 does not change the phase φ. Only a depends on V_2 and the bigger V_2 is the faster a decays with time. □

Remark. If (1.6) is replaced by:

$$E_h = \sum_{j=0}^{N} h^j E_j + O(h^{N+1}) \quad \text{for all } N\in\mathbb{N} \tag{3.5}$$

then we can define:

$$a_j(t,y,\xi) = i\int_0^t \left(\Delta_x a_{j-1}(s,\tilde{x}(s,t,x,\xi),\xi) + \sum_{k=0}^{j-2} E_{j-k}a_k(x,\tilde{x}(x,t,x,\xi),\xi)\right)\eta(s,t,x,\xi)\, ds$$

for all $j \geqslant 2$ and $a \sim \sum h^j a_j$ by Borel theorem (see [EZ, th. 4.16]). Then the rest is of size $O(h^\infty)$ instead of $o(1)$ in (3.4) and hence in (3.16) and (3.26) below.

3.2 Critical points of the phase function

For $t\in[0,3\tau_0]$, $x,\xi\in\mathbb{R}^n$ and $z\in\Gamma_1$ we write:

$$\psi(t,x,z,\xi) = \varphi(t,x,\xi) - \langle z,\xi\rangle$$

In this section we study the critical points of ψ with report to t, ξ and z with $t \in]0, 3\tau_0]$, that is the solutions of the system:

$$\begin{cases} \partial_t \psi(t,x,z,\xi) = 0 \\ \partial_z \psi(t,x,z,\xi) = 0 \\ \partial_\xi \psi(t,x,z,\xi) = 0 \\ t \in]0, 3\tau_0] \end{cases} \iff \begin{cases} \partial_t \varphi(t,x,\xi) = 0 \\ \xi \in N_z\Gamma_1 \\ \partial_\xi \varphi(t,x,\xi) = z \\ t \in]0, 3\tau_0] \end{cases} \tag{3.6}$$

Proposition 3.2. *Let $t \in]0, 3\tau_0]$, $x, \xi \in \mathbb{R}^n$ and $z \in \Gamma$. If (t,x,ξ,z) is a solution of (3.6) then $(z,\xi) \in N_E\Gamma_1$ and $x = \overline{x}(t,z,\xi)$.*

Proof. Assume that (t,x,ξ,z) is such a solution. We already know that $\xi \in N_z\Gamma_1$. By proposition IV.14 in [Rob87] we have:

$$(x, \partial_x \varphi(t,x,\xi)) = \phi^t(\partial_\xi \varphi(t,x,\xi), \xi) = \phi^t(z,\xi) \tag{3.7}$$

and in particular: $x = \overline{x}(t,z,\xi)$. Moreover, since φ is a solution of (3.1) we also have:

$$p(z,\xi) = p(x, \partial_x \varphi(t,x,\xi)) = |\partial_x \varphi(t,x,\xi)|^2 + V_1(x) = E_0 - \partial_t \varphi(t,x,\xi) = E_0$$

which proves that $|\xi|^2 = E_0 - V_1(z)$. ◻

We prove that for x close to Γ (but not on Γ_1), there is a solution (t,x,ξ,z) for (3.6). By proposition 3.2, this solution must be (t_x, x, z_x, ξ_x) (defined in proposition 2.4), so we already have uniqueness.

We consider the function Φ defined as follows: for $y \in \tilde{\Gamma}_1(0, 3\tau_0)$, $\xi \in \mathbb{R}^n$, $\zeta \in T_{z_y}\Gamma_1$ of norm less than 1, $\delta \in [0, \gamma_1]$ (where $\gamma_1 \in]0,1]$ is chosen small enough for $\exp_z(\delta\zeta)$ being defined in Γ_2 for all $z \in \Gamma_1$ and ζ of norm less than 1) and $\theta \in]0, 3\tau_0/\gamma_1]$ then:

$$\Phi(\theta, y, \zeta, \xi, \delta) = \begin{cases} \frac{1}{\delta}\left(\varphi(\delta\theta, \overline{x}(\delta t_y, z_y, \xi_y), \xi) - \left\langle \exp_{z_y}(\delta\zeta), \xi \right\rangle\right) & \text{if } \delta \neq 0 \\ \langle \xi_y - \zeta, \xi \rangle - \theta(\xi^2 + V_1(z_y) - E_0) & \text{if } \delta = 0 \end{cases} \tag{3.8}$$

For $\delta \in]0, \gamma_1]$, $t \in \left]0, \frac{3\tau_0\delta}{\gamma_1}\right]$, $x \in \tilde{\Gamma}_1(0, \delta\tau_0)$, z such that $d_\Gamma(z_x, z) \leqslant \delta$ and $\xi \in \mathbb{R}^n$ we have:

$$\psi(t,x,\xi,z) = \delta\Phi\left(\frac{t}{\delta}, \overline{x}\left(\frac{t_x}{\delta}, z_x, \xi_x\right), \frac{1}{\delta}(\exp_{z_x})^{-1}(z), \xi, \delta\right)$$

Thus:

$$\partial_t \psi(t,x,z,\xi) = 0 \iff \partial_\theta \Phi\left(\frac{t}{\delta}, \overline{x}\left(\frac{t_x}{\delta}, z_x, \xi_x\right), \frac{1}{\delta}(\exp_{z_x})^{-1}(z), \xi\right) = 0 \tag{3.9a}$$

$$\partial_\xi \psi(t,x,z,\xi) = 0 \iff \partial_\xi \Phi\left(\frac{t}{\delta}, \overline{x}\left(\frac{t_x}{\delta}, z_x, \xi_x\right), \frac{1}{\delta}(\exp_{z_x})^{-1}(z), \xi\right) = 0 \tag{3.9b}$$

$$\partial_z \psi(t,x,z,\xi) = 0 \iff \partial_\zeta \Phi\left(\frac{t}{\delta}, \overline{x}\left(\frac{t_x}{\delta}, z_x, \xi_x\right), \frac{1}{\delta}(\exp_{z_x})^{-1}(z), \xi\right) = 0 \tag{3.9c}$$

Proposition 3.3. *Let $K = \mathcal{T}\left(\left[\frac{\tau_0}{2}, 3\tau_0\right] \times \overline{N_E\Gamma}\right)$. There exists $\delta_0 \in]0, \gamma_1]$ such that for all $y \in K$ and $\delta \in [0, \delta_0]$ the system:*

$$\begin{cases} \partial_{\theta,\xi,\zeta}\Phi(\theta, y, \zeta, \xi, \delta) = 0 \\ \theta \in \left]0, \frac{3\tau_0}{\gamma_1}\right] \end{cases} \tag{3.10}$$

has a solution $(\theta, \xi, \zeta) \in]0, \tau_0/\gamma_1] \times \mathbb{R}^n \times T_{z_y}\Gamma$.

Proof. For $\delta \in]0, \gamma_1]$ we compute:

$$\begin{aligned}
\Phi(\theta, y, \zeta, \xi, \delta) &= \frac{1}{\delta}\left(\varphi(\delta\theta, \overline{x}(\delta t_y, z_y, \xi_y), \xi) - \left\langle \exp_{z_y}(\delta\zeta), \xi \right\rangle\right) \\
&= \frac{1}{\delta}\Big(\langle \overline{x}(\delta t_y, z_y, \xi_y), \xi\rangle - \delta\theta(\xi^2 + V_1(\overline{x}(\delta t_y, z_y, \xi_y)) - E_0) \\
&\qquad + \delta^2\theta^2 r(\delta\theta, \overline{x}(\delta t_y, z_y, \xi_y), \xi) - \left\langle \exp_{z_y}(\delta\zeta), \xi \right\rangle \Big) \\
&= \langle 2t_y\xi_y - \zeta, \xi\rangle - \theta\left(\xi^2 + V_1(z_y) - E_0\right) + \theta(V_1(\overline{x}(\delta t_y, z_y, \xi_y)) - V_1(z_y)) \\
&\qquad + \delta\theta^2 r(\delta\theta, \overline{x}(\delta t_y, z_y, \xi_y), \xi, h) - \frac{1}{\delta}\left\langle \exp_{z_y}(\delta\zeta) - z_y - \delta\zeta, \xi \right\rangle \\
&= \langle 2t_y\xi_y - \zeta, \xi\rangle - \theta\left(\xi^2 + V_1(z_y) - E_0\right) + \delta R(\theta, y, \xi, \zeta, \delta)
\end{aligned}$$

where R is of class C^1. This proves that Φ is of class C^1. The point $(\theta, y, \zeta, \xi, 0)$ is a solution of (3.10) if and only if:

$$\begin{cases}
|\xi| = \sqrt{E_0 - V_1(z_y)} \\
\xi \in N^*_{z_y}\Gamma \\
2t_y\xi_y - \zeta = 2\theta\xi \\
\theta \in \left]0, \frac{\tau_0}{\gamma_1}\right]
\end{cases}$$

Let $y \in K$. This system has a unique solution which we denote $(\tilde{\theta}_{y,0}, \tilde{\zeta}_{y,0}, \tilde{\xi}_{y,0})$. It is given by:

$$\tilde{\theta}_{y,0} = t_y; \quad \tilde{\zeta}_{y,0} = 0; \quad \tilde{\xi}_{y,0} = \xi_y \tag{3.11}$$

For $z \in \Gamma$ and $\xi \in \mathbb{R}^n$ we denote by $\xi_z^{/\!/}$ the orthogonal projection of ξ on $T_z\Gamma$ and $\xi_z^{\perp} = \xi - \xi_z^{/\!/}$. Then we have:

$$\operatorname{Hess}_{\theta,\zeta,\xi} \Phi(\theta, y, \zeta, \xi, \delta) = \begin{pmatrix}
0 & 0 & -2\,{}^t\xi_z^{/\!/} & -2\,{}^t\xi_z^{\perp} \\
0 & 0 & -I_d & 0 \\
-2\xi_z^{/\!/} & -I_d & -2\theta I_d & 0 \\
-2\xi_z^{\perp} & 0 & 0 & -2\theta I_{n-d}
\end{pmatrix} + \underset{\delta\to 0}{O}(\delta)$$

and in particular:

$$\det \operatorname{Hess}_{\theta,\zeta,\xi} \Phi(\tilde{\theta}_{y,0}, y, \tilde{\zeta}_{y,0}, \tilde{\xi}_{y,0}, 0) = 2^{n-d+1}(-1)^{n-d} t_y^{n-d-1} |\xi_z|^2$$

The derivative of the function:

$$(\theta, y, \zeta, \xi, \delta) \mapsto \partial_{\theta,\zeta,\xi}\Phi(\theta, y, \zeta, \xi, \delta) \in \mathbb{R}^{n+d+1}$$

with report to θ, ζ and ξ at the point $(\tilde{\theta}_{y,0}, 0, \tilde{\zeta}_{y,0}, \tilde{\xi}_{y,0}, 0)$ is:

$$\operatorname{Hess}_{\theta,\zeta,\xi} \Phi((\tilde{\theta}_{y,0}, 0, \tilde{\zeta}_{y,0}, \tilde{\xi}_{y,0}, 0)) \in \mathrm{GL}_{n+d+1}(\mathbb{R})$$

so we can apply the implicit function theorem around $(\tilde{\theta}_{y,0}, y, \tilde{\zeta}_{y,0}, \tilde{\xi}_{y,0}, 0)$. We obtain that there exists $\delta_y > 0$, a neighborhood $\mathcal{V}_y$ of y in $\mathbb{R}^n$ and a function φ_y which maps $\mathcal{V}_y \times [0, \delta_y]$ into a neighborhood $\mathcal{U}_y$ of $(\tilde{\theta}_{y,0}, \tilde{\zeta}_{y,0}, \tilde{\xi}_{y,0})$ in $]0, \tau_0/\gamma_1] \times T_{z_y}\Gamma \times \mathbb{R}^n$ such that:

$$\forall (v, \delta) \in \mathcal{V}_y \times [0, \delta_y], \forall (\theta, \zeta, \xi) \in \mathcal{U}_y, \quad \partial_{\theta,\zeta,\xi}\Phi(\theta, v, \zeta, \xi, \delta) = 0 \Longleftrightarrow (\theta, \zeta, \xi) = \varphi_y(v, \delta)$$

K is covered by a finite number of such neighborhoods $\mathcal{V}_y$. We get the result if we take for δ_0 the minimum of the corresponding δ_y. ☐

Corollary 3.4. *For all $x \in \tilde{\Gamma}(0, 2\delta_0\tau_0)$ there is a unique $(t, z, \xi) \in]0, \tau_0] \times \Gamma \times \mathbb{R}^n$ such that (t, x, z, ξ) is a solution of the system* (3.6). *Moreover this solution is given by (t_x, x, z_x, ξ_x).*

Proof. After proposition 3.2, there only remains to prove existence. Let $x \in \tilde{\Gamma}(0, 2\delta_0\tau_0)$. There is $\delta \in]0, \delta_0]$ such that $y = \overline{x}\left(\frac{t_x}{\delta}, z_x, \xi_x\right) \in \tilde{\Gamma}(\tau_0, 2\tau_0)$. Proposition 3.3 and equations (3.9) give the result. ☐

3.3 Small times control

We can find a neighborhood $\mathcal{G}$ of $N_E\Gamma_0$ such that for all $t \in [0, \tau_0]$ and $(x, \xi) \in \mathcal{G}$ we have $0 < d_1 \leqslant |\xi| \leqslant d_2$ and $\overline{x}(t, x, \xi) \in \tilde{\Gamma}(2\tau_0)$. We choose a function $\chi \in C_0^\infty(\mathbb{R})$ supported in $]-1, \tau_0[$ and equal to 1 in a neighborhood of 0. For $f \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{G}$, we set:

$$B_0(h) = \frac{i}{h} \int_0^\infty \chi(t) e^{-\frac{it}{h}(H_h - E_h)} Op_h(f) S_h \, dt \tag{3.12}$$

Egorov theorem (see proposition 2.1) yields:

$$\left\| \mathbb{1}_{\mathbb{R}^n \setminus \tilde{\Gamma}(2\tau_0)} B_0(h) \right\|_{L^2(\mathbb{R}^n)} = \underset{h \to 0}{O}(h^\infty) \tag{3.13}$$

Proposition 3.5. *If $\tau_0 > 0$ is small enough, then for all $\varepsilon > 0$, there exists $\tau_1 \in]0, \tau_0]$ and $h_0 > 0$ such that for all $f \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{G}$ we have:*

$$\forall h \in]0, h_0], \quad \left\| \mathbb{1}_{\tilde{\Gamma}(\tau_1)} B_0(h) \right\|_{L^2(\mathbb{R}^n)} \leqslant \varepsilon \tag{3.14}$$

Proof. **1.** If $\mathcal{F}_h$ denotes the semiclassical Fourier transform we have:

$$\begin{aligned}
\mathcal{F}_h S_h(\xi) &= h^{\frac{1-n-d}{2}} \int_{\mathbb{R}^n} \int_\Gamma e^{-\frac{i}{h}\langle x, \xi \rangle} A(z) S\left(\frac{x-z}{h}\right) d\sigma(z) \, dx \\
&= h^{\frac{1+n-d}{2}} \int_\Gamma A(z) e^{-\frac{i}{h}\langle z, \xi\rangle} \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} S(y) \, dy \, d\sigma(z) \\
&= h^{\frac{1+n-d}{2}} \hat{S}(\xi) \int_\Gamma A(z) e^{-\frac{i}{h}\langle z, \xi \rangle} \, d\sigma(z)
\end{aligned}$$

where $\hat{S}$ is the usual Fourier transform of S, and then:

$$\begin{aligned}
Op_h(f) S_h(x) &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x, \xi \rangle} f(x, \xi) \mathcal{F}_h S_h(\xi) \, d\xi \\
&= \frac{h^{\frac{1+n-d}{2}}}{(2\pi h)^n} \int_\Gamma \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x - z, \xi \rangle} A(z) f(x, \xi) \hat{S}(\xi) \, d\xi \, d\sigma(z)
\end{aligned}$$

so:

$$B_0(h) = \frac{ih^{-\frac{1+n+d}{2}}}{(2\pi)^n} \int_0^{+\infty} \int_\Gamma \int_{\mathbb{R}^n} \chi(t) A(z) e^{-\frac{i}{h}\langle z, \xi \rangle} e^{-\frac{it}{h}(H_h - E_h)} \left(e^{\frac{i}{h}\langle \cdot, \xi \rangle} f(\cdot, \xi) \right) \hat{S}(\xi) \, d\xi \, d\sigma(z) \, dt \tag{3.15}$$

Let a and φ given by WKB method (see section 3.1). We define:

$$J(x, h) = \int_0^\infty \int_\Gamma \int_{\mathbb{R}^n} \chi(t) e^{\frac{i}{h}(\varphi(t, x, \xi) - \langle z, \xi \rangle)} a(t, x, \xi, h) A(z) \hat{S}(\xi) \, d\xi \, d\sigma(z) \, dt$$

so that by (3.4):

$$B_0(h) = \frac{ih^{-\frac{1+n+d}{2}}}{(2\pi)^n} J(h) \left(1 + \underset{h \to 0}{o}(1) \right) \quad \text{in } L^2(\mathbb{R}^n) \tag{3.16}$$

Let:

$$\kappa(t, x, z, \xi, h) = \chi(t) a(t, x, \xi, h) A(z) \hat{S}(\xi)$$

κ is smooth and of compact support in t, x, z, ξ so all its derivatives are bounded. We recall that we wrote $\psi(t, x, \xi, z) = \varphi(t, x, \xi) - \langle z, \xi \rangle$.

2. Let $N \in \mathbb{N}$. To estimate J, we define, for all $\delta \in]0, \delta_0]$:

$$J_\delta(x) = \mathbb{1}_{\tilde{\Gamma}(\delta\tau_0, 2\delta\tau_0)}(x) \int_{\mathbb{R}} \int_{\Gamma} \int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(t,x,z,\xi)} \kappa(t,x,z,\xi,h)\, d\xi\, d\sigma(z)\, dt$$

Let:

$$J_\delta^{/\!/}(x) = \mathbb{1}_{\tilde{\Gamma}(\delta\tau_0, 2\delta\tau_0)}(x) \int_{\mathbb{R}} \int_{\Gamma} \int_{\left|\xi_z^{/\!/}\right| > d_1\delta} e^{\frac{i}{h}\psi(t,x,z,\xi)} \kappa(t,x,z,\xi,h)\, d\xi\, d\sigma(z)\, dt$$

Since $\partial_z \psi(t,x,z,\xi) = \xi_z^{/\!/}$, N partial integrations in z show that:

$$\left| J_\delta^{/\!/}(x) \right| \leqslant c\, \mathbb{1}_{\tilde{\Gamma}(\delta\tau_0, 2\delta\tau_0)}(x) \left(\frac{h}{\delta} \right)^N$$

and hence:

$$\left\| J_\delta^{/\!/} \right\|_{L^2(\mathbb{R}^n)} \leqslant c\, h^N\, \delta^{\frac{n-d}{2} - N} \tag{3.17}$$

3. By (3.2) we have:

$$\partial_\xi \psi(t,x,z,\xi) = x - (z + 2t\xi) + t^2 \partial_\xi r(t,x,\xi)$$

and hence:

$$[x - (z + 2t\xi)]^\wedge . \partial_\xi \psi(t,x,z,\xi) = |x - (z + 2t\xi)| + t^2 [x - (z + 2t\xi)]^\wedge . \partial_\xi r(t,x,\xi)$$

where $\hat{x}$ stands for $\frac{x}{|x|}$. For $t \leqslant \delta\tau_0 \min\left(1, \frac{\gamma_m}{4d_2}\right)$ (γ_m is defined in proposition 2.4) and $x \in \tilde{\Gamma}(\delta\tau_0, 2\delta\tau_0)$ we have:

$$|x - (z + 2t\xi)| \geqslant |x - z| - 2t\,|\xi| \geqslant \delta\tau_0\gamma_m - 2td_2 \geqslant \frac{\delta\tau_0\gamma_m}{2}$$

and hence:

$$|x - (z + 2t\xi)| + t^2 [x - (z + 2t\xi)]^\wedge . \partial_\xi r \geqslant \delta \left(\frac{\tau_0\gamma_m}{2} - M\tau_0^2 \right) \tag{3.18}$$

where $M = \|\partial_\xi r\|_{L^\infty([0,\tau_0]\times\mathbb{R}^{2n})}$. Taking τ_0 smaller we may assume that the quantity in brackets is positive.

On the other hand if $t \in \left[\delta \frac{2\tau_0(2d_1+\gamma_M)+1}{d_1}, \tau_0\right]$, z_{xx} is a point of Γ_1 for which $|x - z_{xx}| = d(x, \Gamma_1)$ and $\left|\xi_z^{/\!/}\right| \leqslant \delta d_1$, then:

$$\begin{aligned} |x - (z + 2t\xi)| &\geqslant |z + 2t\xi - z_{xx}| - |x - z_{xx}| \\ &\geqslant \left| z + 2t\xi_z^{\perp} - z_{xx} \right| - 2\delta\tau_0 d_1 - 2\delta\tau_0\gamma_M \\ &\geqslant 2td_1 - 2\delta\tau_0(2d_1 + \gamma_M) \end{aligned}$$

since for t small enough $d(z + 2t\xi_z^{\perp}, \Gamma) = \left|2t\xi_z^{\perp}\right| \geqslant 2t\,|\xi| - 2t\left|\xi_z^{/\!/}\right|$. Thus:

$$|x - (z + 2t\xi)| + t^2 [x - (z + 2t\xi)]^\wedge . \partial_\xi r \geqslant t(d_1 - \tau_0 M) + td_1 - 2\delta\tau_0(2d_1 + \gamma_M) \geqslant \delta + t\frac{d_1}{2} \tag{3.19}$$

if $d_1 \geqslant 2\tau_0 M$, which may be assumed. In particular we have proved that there exists $C, c_0 > 0$ such that:

$$\forall \delta \in]0, \delta_0], \forall t \in \left[0, \frac{\delta}{C}\right] \cup [C\delta, \tau_0], \quad |x - (z + 2t\xi)| + t^2 [x - (z + 2t\xi)]^\wedge . \partial_\xi r \geqslant c_0 \delta$$

on the support of $\mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}(x)\kappa(t,x,z,\xi,h)$. We get:

$$\left|\partial_\xi^\alpha \frac{[x-(z+2t\xi)]^\wedge}{|x-(z+2t\xi)|+t^2[x-(z+2t\xi)]^\wedge.\partial_\xi r(t,x,\xi)}\right| \leqslant c_\alpha \delta^{-|\alpha|}$$

on this support, since the derivatives of $[x-(z+2t\xi)]^\wedge$ with report to ξ are bounded for $t\in[0,\delta/C]\cup[C\delta,\tau_0]$ according to (3.18) and (3.19). We choose a function $\chi_1\in C^\infty(\mathbb{R})$ equal to 1 in a neighborhood of $\left]-\infty,\frac{1}{2C}\right]\cup[2C,+\infty]$ and zero on $\left[\frac{1}{C},C\right]$ and $\chi_0=1-\chi_1$. Then we have $J_\delta = J_\delta^1 + J_\delta^0 + J_\delta^{/\!/}$ where, for $j\in\{0,1\}$:

$$J_\delta^j(x) = \mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}(x)\int_0^\infty \int_\Gamma \int_{\left|\xi_z^{/\!/}\right|\leqslant \delta d_1} \chi_j\left(\frac{t}{\delta}\right) e^{\frac{i}{h}\psi(t,x,z,\xi)}\kappa(t,x,z,\xi,h)\,d\xi\,d\sigma(z)\,dt$$

We consider the operator:

$$L : u\mapsto \left((t,x,z,\xi,h)\mapsto -ih\frac{[x-(z+2t\xi)]^\wedge.\partial_\xi u}{|x-(z+2t\xi)|+t^2[x-(z+2t\xi)]^\wedge.\partial_\xi r}\right)$$

The function $(t,x,z,\xi,h)\mapsto \exp\left(\frac{i}{h}\psi(t,x,z,\xi)\right)$ is invariant by L and the adjoint L^* is given by:

$$L^* : v\mapsto \left((t,x,z,\xi)\mapsto ih\,\mathrm{div}_\xi\left(\frac{[x-(z+2t\xi)]^\wedge.v}{|x-(z+2t\xi)|+t^2[x-(z+2t\xi)]^\wedge.\partial_\xi r}\right)\right)$$

N partial integrations with L prove:

$$\left|J_\delta^1(x)\right| \leqslant C_N\left(\frac{h}{\delta}\right)^N \mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}(x)$$

and hence:

$$\left\|J_\delta^1\right\| \leqslant C_N h^N \delta^{\frac{n-d}{2}-N} \tag{3.20}$$

4. We now turn to J_δ^0. We recall that for all $z\in\Gamma_1$ and $\zeta\in T_z\Gamma_1$ of norm less than γ_1 then $\exp_z(\zeta)$ is well-defined (on Γ_2) and $d_{\Gamma_1}(z,\exp_z(\zeta))=|\zeta|$. For τ_0 small enough, if $x\in\tilde\Gamma(\delta\tau_0,2\delta\tau_0)$ and $d_{\Gamma_1}(z,z_x)\geqslant\gamma_1\delta$ then $|x-z|\geqslant\frac{\gamma_1\delta}{2}$ and $|x-(z+2t\xi)|\geqslant\frac{\gamma_1\delta}{4}$. As a result we can do partial integrations with L as before and see that modulo $O((h/\delta)^N)$, $J_\delta^0(x)$ is given by integration over z in a neighborhood of radius δ around z_x:

$$\begin{aligned}J_\delta^0(x) = \mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}(x)\int_0^\infty\!\!\int_{B_\Gamma(z_x,\gamma_1\delta)}\int_{\left|\xi_z^{/\!/}\right|\leqslant\delta d_1}\chi_0\left(\frac{t}{\delta}\right)e^{\frac{i}{h}\psi(t,x,z,\xi)}\kappa(t,x,z,\xi,h)\,d\xi d\sigma(z)\,dt\\ + O\left((h/\delta)^N\right)\end{aligned}$$

After the change of variables $t=\theta\delta$ and $z=\exp_{z_x}(\delta\zeta)$, $\zeta\in T_{z_x}\Gamma$, we get for $y\in\mathbb{R}^n$:

$$\begin{aligned}J_\delta^0(\overline{x}(\delta t_y,z_y,\xi_y)) = \delta^{1+d}\mathbb{1}_{\tilde\Gamma(\tau_0,2\tau_0)}(y)\iint_{\mathbb{R}^n}\int_0^\infty \chi_0(\theta)\tilde\kappa(\theta,y,\xi,\zeta,h)e^{\frac{i}{h}\delta\Phi(\theta,y,\xi,\zeta,\delta)}\,d\theta\,d\xi\,d\zeta\\ + O\left((h/\delta)^N\right)\end{aligned}$$

where integral in ζ is over the ball or radius γ_1 in $T_{z_y}\Gamma$ and:

$$\tilde\kappa(\theta,y,\xi,\zeta,h,\delta) = \tilde\chi(y)\kappa(\delta\theta,\delta y,\xi,\exp_{z_x}(\delta\zeta),h)\partial_\zeta\exp_{z_x}(\delta\zeta)$$

with $\tilde\chi\in C_0^\infty(\mathbb{R}^{2n})$ supported in $\{\tau_0/2\leqslant t_y\leqslant 3\tau_0\}$ and equal to 1 on $\{\tau_0\leqslant t_y\leqslant 2\tau_0\}$. $\tilde\kappa(h,\delta)$ is of compact support in $]0,+\infty[\times(\mathbb{R}^n\setminus\Gamma)\times(\mathbb{R}^n\setminus\{0\})\times T_{z_y}\Gamma$. Φ is defined in (3.8). For y such that $\tau_0/2\leqslant t_y\leqslant 3\tau_0$ and $\delta\in]0,\delta_0]$, there is by proposition 3.3 a unique $(\tilde\theta_{y,\delta},\tilde\xi_{y,\delta},\tilde\zeta_{y,\delta})$ such that $(\tilde\theta_{y,\delta},y,\tilde\xi_{y,\delta},\tilde\zeta_{y,\delta},\delta)$ is a critical point of ϕ and $\tilde\theta>0$. Moreover:

$$\begin{aligned}\partial_{\theta,\xi,z}\Phi(\theta,y,z,\xi,\delta) = \mathrm{Hess}_{\theta,z,\xi}\,\Phi(\tilde\theta_{y,\delta},y,\tilde\zeta_{y,\delta},\tilde\xi_{y,\delta},\delta)((\theta,z,\xi)-(\tilde\theta_{y,\delta},\tilde\zeta_{y,\delta},\tilde\xi_{y,\delta}))\\ + \underset{(\theta,\zeta,\xi)\to(\tilde\theta_{y,\delta},\tilde\zeta_{y,\delta},\tilde\xi_{y,\delta})}{O}(|\theta-\tilde\theta_{y,\delta}|,|\zeta-\tilde\zeta_{y,\delta}|,|\xi-\tilde\xi_{y,\delta}|)\end{aligned}$$

and hence:

$$(\theta,\zeta,\xi)-(\tilde\theta_{y,\delta},\tilde\zeta_{y,\delta},\tilde\xi_{y,\delta}) = \left[\mathrm{Hess}_{\theta,\zeta,\xi}\,\Phi(\tilde\theta_{y,\delta},y,\tilde\zeta_{y,\delta},\tilde\xi_{y,\delta},\delta)\right]^{-1}(\partial_{\theta,\zeta,\xi}\Phi(\theta,\zeta,\xi)) \\ + \underset{(\theta,\zeta,\xi)\to(\tilde\theta_{y,\delta},\tilde\zeta_{y,\delta},\tilde\xi_{y,\delta})}{O}(|\theta-\tilde\theta_{y,\delta}|,|\zeta-\tilde\zeta_{y,\delta}|,|\xi-\tilde\xi_{y,\delta}|)$$

y and δ stay in a compact set and zero is never an eigenvalue of $\mathrm{Hess}_{\theta,\zeta,\xi}\,\Phi(\tilde\theta_{y,\delta},y,\tilde\zeta_{y,\delta},\tilde\zeta_{y,\delta},\delta)$, so the norm of $\mathrm{Hess}_{\theta,\zeta,\xi}(\theta,\zeta,\xi)^{-1}$ is bounded.

As a consequence the quantity:

$$\frac{\left|(\theta,\zeta,\xi)-(\tilde\theta_{y,\delta},\zeta_{y,\delta},\tilde\xi_{y,\delta})\right|}{|\partial_{\theta,\zeta,\xi}\Phi(\theta,y,z,\xi,\delta)|}$$

is uniformly bounded. So we can use theorems 7.7.5 and 7.7.6 in [Hör84], which give:

$$\left|J^0_\delta(x)\right| \leqslant c\delta^{1+d}\left(\frac{h}{\delta}\right)^{\frac{n+d+1}{2}}\mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}(x)+c\left(\frac{h}{\delta}\right)^N\mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}(x)$$

and thus:

$$\left\|J^0_\delta\right\| \leqslant c\delta^{\frac12}h^{\frac{n+d+1}{2}}+c\,h^N\delta^{\frac{n-d}{2}-N} \tag{3.21}$$

5. For $\gamma\in]0,1]$ we define :

$$\tilde J_\gamma(x)=\mathbb{1}_{\tilde\Gamma(2\gamma\tau_0)}(x)\int_{\mathbb{R}}\int_\Gamma\int_{\mathbb{R}^n}e^{\frac{i}{h}\psi(t,x,z,\xi)}\kappa(t,x,z,\xi,h)\,d\xi\,d\sigma(z)\,dt$$

$\tilde J^{/\!/}_\gamma$ is defined as $J^{/\!/}_\delta$ with $\mathbb{1}_{\tilde\Gamma(\delta\tau_0,2\delta\tau_0)}$ replaced by $\mathbb{1}_{\tilde\Gamma(2\gamma\tau_0)}$. An estimate analog to (3.17) holds for $\tilde J^{/\!/}_\gamma$. We now note $\chi_+=\mathbb{1}_{[C,+\infty[}\chi_1$, $\chi_-=1-\chi_+$, and:

$$\tilde J^\pm_\gamma(x)=\mathbb{1}_{\tilde\Gamma(2\gamma\tau_0)}(x)\int_{\mathbb{R}}\int_\Gamma\int_{\left|\xi_z^{/\!/}\right|\leqslant\gamma d_1}\chi_\pm\left(\frac{t}{\gamma}\right)e^{\frac{i}{h}\psi(t,x,z,\xi)}\kappa(t,x,z,\xi,h)\,d\sigma(z)\,d\xi\,dt$$

As we did for J^1_δ we see that:

$$\left\|\tilde J^+_\gamma\right\| \leqslant C_N h^N\gamma^{\frac{n-d}{2}-N} \tag{3.22}$$

To estimate J^-_γ, we remark that we are integrating a bounded function over a set of size $O(\gamma)$ in t and over $\{(z,\xi),\left|\xi_z^{/\!/}\right|\leqslant\gamma d_1\}$ whose volume is of size $O(\gamma^d)$, so:

$$\left|\tilde J^-_\gamma(x)\right| \leqslant c\gamma^{1+d}\mathbb{1}_{\tilde\Gamma(2\gamma\tau_0)}(x)$$

Taking the $L^2(\mathbb{R}^n)$ norm in x gives:

$$\left\|\tilde J^-_\gamma\right\|_{L^2(\mathbb{R}^n)} \leqslant c\gamma^{1+\frac{n+d}{2}} \tag{3.23}$$

6. Estimates (3.17), (3.20), (3.21), (3.22) and (3.23) allow to conclude: let $\tau_1\in]0,\delta_0\tau_0]$ and $\mu\in]0,1[$, we use a dyadic decomposition $\delta=2^{-m}$ with $h^{1-\mu}<\delta<\tau_1/\tau_0$, that is $\ln_2(\tau_0)-\ln_2(\tau_1)<m<-(1-\mu)\ln_2 h$. We write $m_-=\ln_2(\tau_0)-\ln_2(\tau_1)$ and $m_+=-(1-\mu)\ln_2 h$. Then:

$$\left\|\mathbb{1}_{\tilde\Gamma(\tau_1)}J\right\| \leqslant \left\|\tilde J_{h^{1-\mu}}\right\|+\sum_{m_-<m<m_+}\|J_{2^{-m}}\|$$

with:

$$\begin{aligned}\left\|\tilde J_{h^{1-\mu}}\right\| &\leqslant \left\|\tilde J^{/\!/}_{h^{1-\mu}}\right\|+\left\|\tilde J^-_{h^{1-\mu}}\right\|+\left\|\tilde J^+_{h^{1-\mu}}\right\| \\ &\leqslant c_N\left(h^{(1-\mu)\left(\frac{n+d}{2}+1\right)}+h^{(1-\mu)\frac{n-d}{2}+\mu N}\right) \\ &\leqslant c_N h^{\frac{n+d+1}{2}}\left(h^{\frac12-\mu\left(\frac{n+d}{2}+1\right)}+h^{\mu N-\frac12-d-\mu\frac{n-d}{2}}\right)\end{aligned}$$

and:

$$\begin{aligned}
\sum_{m_-<m<m_+} \|J_{2^{-m}}\| &\leqslant \sum_{m_-<m<m_+} \left(\left\|J^1_{2^{-m}}\right\| + \left\|J^0_{2^{-m}}\right\| + \left\|J^{/\!/}_{2^{-m}}\right\| \right) \\
&\leqslant c_N \left(h^N \sum_{m \leqslant m_+} \left(2^{N-\frac{n-d}{2}}\right)^m + h^{\frac{n+d+1}{2}} \sum_{m_- \leqslant m} 2^{-\frac{m}{2}} \right) \\
&\leqslant c_N \left(h^{N-(1-\mu)\left(N-\frac{n-d}{2}\right)} + h^{\frac{n+d+1}{2}} \sqrt{\tau_1} \right) \\
&\leqslant c_N h^{\frac{n+d+1}{2}} \left(h^{\mu N - \frac{1}{2} - d - \mu\frac{n-d}{2}} + \sqrt{\tau_1} \right)
\end{aligned}$$

We now take $\mu > 0$ small enough to have $\nu := \frac{1}{2} - \mu\left(\frac{n+d}{2} + 1\right) > 0$ and then N big enough to have $\mu N - \frac{1}{2} - d - \mu\frac{n-d}{2} \geqslant 0$. This gives:

$$\left\| \mathbb{1}_{\tilde{\Gamma}(\tau_1)} J \right\|_{L^2(\mathbb{R}^n)} \leqslant c\, h^{\frac{n+d+1}{2}} (\sqrt{\tau_1} + h^\nu)$$

If τ_1 and h_0 are small enough we have $c(\sqrt{\tau_1} + h^\nu) \leqslant \frac{\varepsilon}{2}$ for all $h \in]0, h_0]$. By (3.16), if h_0 is small enough we finally reach the result:

$$\left\| \mathbb{1}_{\tilde{\Gamma}(\tau_1)} B_0(h) \right\|_{L^2(\mathbb{R}^n)} \leqslant \varepsilon$$

□

For $z \in \Gamma$ and $x \in \mathbb{R}^n$ we set:

$$\tilde{\psi}_{x,z} : (t, \zeta, \xi) \mapsto \psi(t, x, \exp_z(\zeta), \xi) \tag{3.24}$$

This is defined for $t \in]0, \tau_0]$, $\xi \in \mathbb{R}^n$ and ζ in a neighborhood $\mathcal{U}_z$ of 0 in $T_z\Gamma$. Now for $x \in \tilde{\Gamma}(0, 2\tau_0)$ we let $\psi(x) = \psi(t_x, x, z_x, \xi_x) = \varphi(t_x, x, \xi_x) - \langle z_x, \xi_x \rangle$ and:

$$b_0(x) = i(2\pi)^{\frac{d+1-n}{2}} \frac{e^{\frac{i\pi}{4} \operatorname{sgn} \operatorname{Hess} \tilde{\psi}_{x,z_x}(t_x, 0, \xi_x)}}{\left|\det \operatorname{Hess} \tilde{\psi}_{z_x}(t_x, 0, \xi_x)\right|^{\frac{1}{2}}} A(z_x) a_0(t_x, x, \xi_x) \hat{S}(\xi_x) \chi(t_x) \tag{3.25}$$

Proposition 3.6. *Let $\mathcal{U}$ be a neighborhood of Γ_0 in $\mathbb{R}^n$. Then on $\tilde{\Gamma}(\tau_0) \setminus \mathcal{U}$ the function B_0 is a lagrangian distribution of phase ψ and principal symbol b_0.*

This means that B_0 is of the form $B_0(x) = e^{\frac{i}{h}\psi(x)} b_0(x) + o(1)$. Note that if (3.5) holds we can have $B_0(x) = e^{\frac{i}{h}\psi(x)} b(x, h) + O(h^\infty)$ where $b(x, h) \sim \sum h^j b_j(x)$ for some functions b_j, $j \geqslant 1$. See [Sog02] for more details about lagrangian distributions (in the microlocal setting).

Proof. Everything we need is already in the proof of proposition 3.5. By Egorov theorem there exists $\tau_2 \in]0, \tau_0]$ such that:

$$\mathbb{1}_{\tilde{\Gamma}(\tau_0) \setminus \mathcal{U}} B_0 = \mathbb{1}_{\tilde{\Gamma}(\tau_2, \tau_0)} B_0 + \underset{h \to 0}{O}(h^\infty)$$

Let us come back to the proof of (3.18) with $\delta = \tau_2$. We see that if $\overline{\chi} \in C_0^\infty(\mathbb{R}_+^*)$ is such that $\overline{\chi}(t) = \chi(t)$ for $t \geqslant \frac{\gamma_m \tau_2 \tau_0}{4 d_2}$ then in $L^2(\tilde{\Gamma}(\tau_2, \tau_0))$:

$$B_0(x) = \frac{i h^{-\frac{1+n+d}{2}}}{(2\pi)^n} \int_0^\infty \int_\Gamma \int_{\mathbb{R}^n} \overline{\chi}(t) e^{\frac{i}{h}\psi(t,x,z,\xi)} a(t, x, \xi, h) A(z) \hat{S}(\xi)\, d\xi\, d\sigma(z)\, dt \left(1 + \underset{h \to 0}{o}(1)\right)$$

Moreover as we explained for J^0_δ the only relevant part of integration on z is around z_x, so:

$$\begin{aligned}
B_0(x, h) = \frac{i h^{-\frac{1+n+d}{2}}}{(2\pi)^n} \int_0^\infty \int_{\mathcal{U}_{z_x}} \int_{\mathbb{R}^n} \overline{\chi}(t) e^{\frac{i}{h}\tilde{\psi}_{x,z_x}(t,\zeta,\xi)} a(t, x, \xi, h) A(z) \hat{S}(\xi) \operatorname{Jac}(\exp_{z_x})(\zeta)\, d\xi\, d\zeta\, dt \\
\times \left(1 + \underset{h \to 0}{o}(1)\right)
\end{aligned} \tag{3.26}$$

Then, as we did to study J_δ^0, we use the results of section 3.2 and stationnary phase method to get the result (in particular the only stationnary point for $\tilde{\psi}_{x,z_x}$ is $(t_x, 0, \xi_x)$. $\square$

Proposition 3.7. *Let $x \in \tilde{\Gamma}(\tau_0)$. We have:*

$$\left|\det \operatorname{Hess} \tilde{\psi}_{x,z_x}(t_x, 0, \xi_x)\right| = 2^{n-d+1} t_x^{n-d-1} \left|\xi_x\right|^2 + \underset{t_x \to 0}{O}(t_x^{n-d}) \tag{3.27}$$

where the size of the rest is uniform in x.

Proof. (ii). By (3.2) we have:

$$\begin{aligned} \det \operatorname{Hess} \tilde{\psi}_{x,z}(t,0,\xi) &= \begin{vmatrix} \partial_t^2 \varphi(t,x,\xi) & 0 & -2\,{}^t\xi_z^{/\!/} & -2\,{}^t\xi_z^{\perp} \\ 0 & A & -I_d & 0 \\ -2\xi_z^{/\!/} & -I_d & -2tI_d & 0 \\ -2\xi_z^{\perp} & 0 & 0 & -2tI_{n-d} \end{vmatrix} \left(1 + \underset{t\to 0}{O}(t)\right) \\ &= (-1)^{n-d} 2^{n-d+1} t^{n-d-1} \left|\xi_z^{\perp}\right|^2 + \underset{t \to 0}{O}(t^{n-d}) \end{aligned}$$

where for $1 \leqslant i, j \leqslant d$:

$$A_{ij} = -\partial^2_{\zeta_i \zeta_j} \left\langle \exp_z(\zeta), \xi \right\rangle$$

only appears in the rest, and $(\xi_x)_{z_x}^{\perp} = \xi_x$ since $(z_x, \xi_x) \in N_E\Gamma$. $\square$

4 Partial result for finite times

4.1 Intermediate times contribution

We begin with a proposition which proves that for $w \in \mathbb{R}^{2n}$ and $q \in C_0^\infty(\mathbb{R}^{2n})$ supported close to w, then in the integral:

$$u_h^T = \frac{i}{h} \int_0^T U_h^E(t) S_h \, dt$$

only times around $t_{w,k}$ for $1 \leqslant k \leqslant K_w^T$ (and on a neighborhood of 0 if $w \in N_E\Gamma$) give a relevant contribution.

Proposition 4.1. *Let $w \in \mathbb{R}^{2n}$, $T > 0$ and $\tilde{\chi} \in C_0^\infty(\mathbb{R})$ a function which is zero near $t_{w,k}$ for $k \in [\![1, K_w]\!]$ (and 0 if $w \in N_E\Gamma$). Then there exists a neighborhood $\mathcal{V}_{w,T}$ of w in $\mathbb{R}^{2n}$ and a neighborhood $\mathcal{G}_{w,T} \subset \mathcal{G}$ of $N_E\Gamma$ ($\mathcal{G}$ was defined in section 3.3) such that for all $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_{w,T}$ and $f \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{G}_{w,T}$, we have in $L^2(\mathbb{R}^n)$:*

$$Op_h^w(q) \left(\frac{i}{h} \int_0^T \tilde{\chi}(t) U_h^E(t) Op_h(f) S_h \, dt \right) = \underset{h \to 0}{O}(h^\infty)$$

Proof. There exists a neighborhood $\mathcal{G}_{w,T} \subset \mathcal{G}$ of $N_E\Gamma$ in $\mathbb{R}^{2n}$ and $\rho > 0$ such that for all $\tilde{w} \in \mathcal{G}$ and $t \in \operatorname{supp} \tilde{\chi}$ we have:

$$\left|\phi^t(\tilde{w}) - w\right| \geqslant 2\rho$$

Otherwise for all $m \in \mathbb{N}^*$ we can find $t_m \in \operatorname{supp} \tilde{\chi}$ and $w_m \in \mathbb{R}^{2n}$ with $d(w_m), N_E\Gamma) \leqslant \frac{1}{m}$ such that $|\phi^{t_m}(w_m) - w| \leqslant \frac{1}{m}$. We can extract a subsequence so that $t_{m_k} \to t \in \operatorname{supp} \tilde{\chi}$ and $w_{m_k} \to w_\infty \in N_E\Gamma$. Then we have $\phi^t(w_\infty) = w$, which is impossible since $t \notin \{t_{w,1}, \dots, t_{w,K_w}\}$ ($\cup \{0\}$ if $w \in N_E\Gamma$).

Let $\mathcal{V}_{w,T}$ be the ball $B(w, \rho)$ and $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_{w,T}$. By Egorov theorem, we have for all $t \in [0, T]$:

$$\left\| Op_h^w(q) U_h^E(t) Op_h(f) \right\| = \underset{h \to 0}{O}(h^\infty)$$

where the remainder is uniform in $t \in [0, T]$. An integration over t gives the result. $\square$

Remark. Note that neither the neighborhoods $\mathcal{G}_{w,T}$ and $\mathcal{V}_{w,T}$ nor the size of the remainder can be uniform in T. That is the main reason why we cannot deal directly with u_h and have to begin with a study of u_h^T.

Let $w \in \Lambda$ and $\tau_w = \min(t_{w,1}, \tau_0)$. We consider $\chi_w \in C_0^\infty(\mathbb{R})$ supported in $]0, 2\tau_w[$ and equal to 1 in a neighborhood of τ_w, and set:

$$B_w(h) = \frac{i}{h} \int_{t=0}^{\infty} \chi_w(t) U_h^E(t) Op_h(f) S_h \, dt$$

Moreover, for $k \in [\![1, K_w]\!]$ we denote:

$$B_{w,k}(h) = \frac{i}{h} \int_{t=0}^{\infty} \chi_w(t - t_{w,k} + \tau_w) U_h^E(t) Op_h(f) S_h \, dt \tag{4.1}$$

As in proposition 3.6 (and we do not even have to worry about very small times since χ_w vanishes around 0) we see that $B_w(h)$ is a lagrangian distribution of submanifold

$$\begin{aligned}
\Lambda_0 &= \left\{ (x, \partial_x \psi), x \in \tilde{\Gamma}(0, 2\tau_0) \right\} = \left\{ \phi^{t_x}(z_x, \xi_x), x \in \tilde{\Gamma}(0, 2\tau_0) \right\} \\
&= \left\{ \phi^t(z, \xi), t \in]0, 2\tau_0], (z, \xi) \in N_E \Gamma \right\}
\end{aligned}$$

and of principal symbol

$$b_w(x) = i(2\pi)^{\frac{d+1-n}{2}} \frac{e^{\frac{i\pi}{4} \operatorname{sgn} \operatorname{Hess} \tilde{\psi}_{x,z_x}(t_x, 0, \xi_x)}}{\left| \det \operatorname{Hess} \tilde{\psi}_{z_x}(t_x, 0, \xi_x) \right|^{\frac{1}{2}}} A(z_x) a_0(t_x, x, \xi_x) \hat{S}(\xi_x) \chi_w(t_x)$$

Proposition 4.2. *For all $w \in \Lambda$ and $k \in [\![1, K_w]\!]$, $B_{w,k}(h)$ is a lagrangian distribution of lagrangian submanifold $\Lambda_{w,k} := \phi^{t_{w,k}} \Lambda_0$. We denote by $b_{w,k}$ and $\psi_{w,k}$ the principal symbol and the phase of this distribution.*

Remark. Again, with (1.6) this means that $B_{w,k}(h) = e^{\frac{i}{h}\psi_{w,k}} b_{w,k} + o(1)$, but with assumption (3.5) we can write $B_{w,k}(h) = e^{\frac{i}{h}\psi_{w,k}} \tilde{b}_{w,k}(h) + O(h^\infty)$ where $\tilde{b}_{w,k}(h)$ is a classical symbol of principal symbol $b_{w,k}$.

Proof. We have:

$$\begin{aligned}
B_{w,k}(h) &= \frac{i}{h} \int_{t=0}^{\infty} \chi_w(t - t_{w,k} + \tau_w) U_h^E(t) Op_h(f) S_h \, dt \\
&= \frac{i}{h} \int_{t=-t_{w,k}+\tau_w}^{\infty} \chi_w(t) U_h^E(t + t_{w,k} - \tau_w) Op_h(f) S_h \, dt \\
&= U_h^E(t_{w,k} - \tau_w) B_w(h)
\end{aligned}$$

It is known that $e^{-\frac{i(t_{w,k}-\tau_w)}{h}(H_1^h - E_h)}$ turns a lagrangian distribution of submanifold Λ_0 into a lagrangian distribution of submanifold $\phi^{t_{w,k}-\tau_w}\Lambda_0$ (see [Sog02, EZ]). We can similarly see that this also applies to $U_h^E(t_{w,k} - \tau_w)$. Computations are actually close to what is done for WKB method, where we see that the imaginary part does not affect the phase factor but only the amplitude. Here again V_2 only appears in the symbol $b_{w,k}$ of the lagrangian distribution. □

We give another property of $B_{w,k}$ we are going to use in section 5.3:

Proposition 4.3. *Let $w \in \Lambda$. For all $k \in [\![1, K_w]\!]$ we have:*

$$(H_h - E_h) B_{w,k}(h) = 0 \quad \textit{microlocally near } w$$

Proof. We have:

$$\begin{aligned}(H_h - E_h)B_{w,k}(h) &= (H_h - E_h)\frac{i}{h}\int_0^{+\infty} \chi_w(t - t_k + \tau_w)U_h^E(t)Op_h(f)S_h\,dt\\ &= -\int_0^{+\infty} \chi_w(t - t_k + \tau_w)\partial_t U_h^E(t)Op_h(f)S_h\,dt\\ &= \int_0^{+\infty} \chi_w'(t - t_k + \tau_w)U_h^E(t)Op_h(f)S_h\,dt\end{aligned}$$

As $\partial_t\chi_w(t - t_k + \tau_w)$ is zero near $t = t_j$ for $j \in [\![1, K_w]\!]$ (and $t = 0$), the result is given by Egorov theorem as in the proof of theorem 4.1. $\square$

4.2 Convergence toward a partial semiclassical measure

We are now ready to give the semiclassical measure for u_h^T.

Theorem 4.4. *Let $T \geqslant 0$. There exists a nonnegative Radon measure μ_T on $\mathbb{R}^{2n}$ such that for all $q \in C_0^\infty(\mathbb{R}^{2n})$ we have:*

$$\left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle \xrightarrow[h\to 0]{} \int q\,d\mu_T \tag{4.2}$$

Proof. **1. Localization around a point $w \in \mathbb{R}^{2n}$.** We are going to show that for any $w \in \mathbb{R}^{2n}$ and $T \geqslant 0$, there is a neighborhood $\mathcal{V}_{w,T} \subset \mathbb{R}^{2n}$ such that for all $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_{w,T}$ we have:

$$\left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle \xrightarrow[h\to 0]{} \int q\,d\mu_{w,T} \tag{4.3}$$

where $\mu_{w,T}$ is a Radon measure on $\mathcal{V}_{w,T}$. If $w_1, w_2 \in \mathbb{R}^{2n}$ are such that $\mathcal{V}_{w_1,T} \cap \mathcal{V}_{w_2,T} \neq \emptyset$, then the two measures $\mu_{w_1,T}$ and $\mu_{w_2,T}$ coincide on $\mathcal{V}_{w_1,T} \cap \mathcal{V}_{w_2,T}$ (we only have to consider the two versions of (4.3) for $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_{w_1,T}\cap\mathcal{V}_{w_2,T}$). Thus we can define the measure μ_T on $\mathbb{R}^{2n}$ as the only measure which coincides with $\mu_{w,T}$ on $\mathcal{V}_{w,T}$ for all $w \in \mathbb{R}^{2n}$. Then for all $q \in C_0^\infty(\mathbb{R}^{2n})$ a partition of unity and a finite numbers of applications of (4.3) give (4.2).

So let $w \in \mathbb{R}^{2n}$. If $w \notin (N_E\Gamma \cup \Lambda)$ we can choose a neighborhood $\mathcal{V}_w$ of w which does not intersect $N_E\Gamma \cup \Lambda$. Proposition 4.1 with $\tilde{\chi} = 1$ on $[0, T]$ shows:

$$\left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle \xrightarrow[h\to 0]{} 0$$

for all $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_w$. Hence we set $\mu_{w,T} = 0$ on $\mathcal{V}_{w,T}$. This proves that if μ_T exists then we must have:

$$\mu_T = 0 \quad \text{outside } N_E\Gamma \cup \Lambda \tag{4.4}$$

We now assume that $w \in N_E\Gamma \cup \Lambda$.

2. Localization around relevant times. Let $\delta_w = 1$ if $w \in N_E\Gamma$ and $\delta_w = 0$ otherwise. We recall that χ and χ_w have been chosen in sections 3.3 and 4.1. By corollary 2.5, if $w \in N_E\Gamma$ then $t_{w,1} \geqslant 3\tau_0$ so for all $w \in N_E\Gamma \cup \Lambda$ supports of functions $\delta_w\chi$ and $\chi_w(\cdot - t_{w,k} + \tau_w)$ for $1 \leqslant k \leqslant K_w^T$ are pairwise disjoint, so we can consider a function $\tilde{\chi} \in C_0^\infty(\mathbb{R}, [0, 1])$ such that:

$$\forall t \in [0, T], \quad \delta_w\chi(t) + \sum_{k=1}^{K_w^T} \chi_w(t - t_k + \tau_w) + \tilde{\chi}(t) = 1$$

By proposition 4.1 there exists a function $f_{w,T} \in C_0^\infty(\mathbb{R}^{2n})$ equal to 1 around $N_E\Gamma$ and a neighborhood $\mathcal{V}_{w,T}$ of w in $\mathbb{R}^{2n}$ such that for q supported in $\mathcal{V}_{w,T}$ we have in $L^2(\mathbb{R}^n)$:

$$Op_h^w(q)v_h^T = Op_h^w(q)\tilde{u}_h^T + \underset{h\to 0}{O}(h^\infty)$$

where:

$$v_h^T = \frac{i}{h}\int_0^T U_h^E(t)Op_h(f_{w,T})S_h\,dt \quad \text{and} \quad \tilde{u}_h^T = \delta_w B_{w,0}^T + \sum_{k=1}^{K_w^T} B_{w,k}^T$$

with $B_{w,0}^T$ defined in (3.12) and the $B_{w,k}^T$ given by (4.1) with f replaced by $f_{w,T}$. Let $\tilde{g}$ be given by proposition 2.10. We have:

$$\begin{aligned}
&\left\langle Op_h^w(q)\tilde{u}_h^T, \tilde{u}_h^T\right\rangle \qquad (4.5)\\
&\quad = \left\langle Op_h^w(q)\left(v_h^T + (1-\tilde{g})(H_1^h)(u_h^T - v_h^T) + O(h)\right), v_h^T + (1-\tilde{g})(H_1^h)(u_h^T - v_h^T) + O(h)\right\rangle\\
&\quad = \left\langle Op_h^w(q)v_h^T, v_h^T\right\rangle + \left\langle Op_h^w(q)(u_h^T - v_h^T), (1-\tilde{g})(H_1^h)v_h^T\right\rangle\\
&\qquad + \left\langle Op_h^w(q)(1-\tilde{g})(H_1^h)v_h^T, u_h^T - v_h^T\right\rangle + \underset{h\to 0}{O}(\sqrt{h})\\
&\quad = \left\langle Op_h(q)\tilde{u}_h^T, \tilde{u}_h^T\right\rangle + \underset{h\to 0}{O}(\sqrt{h})
\end{aligned}$$

3. Definition of the measure $\mu_{w,T}$. For $k \in [\![1, K_w^T]\!]$ and Ω a borelian set in $\mathcal{V}_{w,T}$ we define:

$$\mu_{w,T,k}(\Omega) = \int_{\mathbb{R}^n} \mathbb{1}_\Omega(x, \partial\psi_{w,k}(x))\,|b_{w,k}(x)|^2\,dx\,; \quad \mu_{w,T,0}(\Omega) = \delta_w \int_{\mathbb{R}^n} \mathbb{1}_\Omega(x, \partial\psi(x))\,|b_0(x)|^2\,dx$$

and finally:

$$\mu_{w,T}(\Omega) = \sum_{k=0}^{K_w^T} \mu_{w,T,k}$$

which defines a measure on $\mathcal{V}_{w,T}$. Note that all these measures are nonnegative. $\mathcal{V}_{w,T}$ and $\mu_{w,T}$ are now fixed, and we have to prove that for any $\varepsilon > 0$ and $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_{w,T}$, there is $h_0 > 0$ such that for all $h \in]0, h_0]$:

$$\left|\left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle - \int q\,d\mu_{w,T}\right| \leqslant \varepsilon \tag{4.6}$$

Let $\varepsilon > 0$ and q supported in $\mathcal{V}_{w,T}$. (4.5) yields:

$$\left|\left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle - \left\langle Op_h^w(q)\tilde{u}_h^T, \tilde{u}_h^T\right\rangle\right| \leqslant \frac{\varepsilon}{9} \tag{4.7}$$

with $h \in]0, h_0]$ for some $h_0 > 0$.

4. Self-intersections of Λ. Let $j, k \in [\![1, K_w]\!]$ with $j \neq k$ ($j, k \in [\![0, K_w]\!]$ if $w \in N_E\Gamma$). $\Lambda_{w,j} \cap \Lambda_{w,k}$ is a closed set of measure 0 in the smooth manifold $\Lambda_{w,j}$, hence by regularity of the measure on $\Lambda_{w,j}$, for all $m \in \mathbb{N}$ we can find an open subset U_j^m of $\Lambda_{w,j}$ of measure less than $\frac{1}{m}$ such that $\Lambda_{w,j} \cap \Lambda_{w,k} \subset U_j^m$. We can find an open sett V_j^m in $\mathbb{R}^{2n}$ of measure less than $\frac{1}{m}$ such that $U_j^m = V_j^m \cap \Lambda_{w,j}$, and by Uryshon lemma there exists a function $\gamma_j^m \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ equal to 1 outside V_j^m and zero in a neighborhood of $\Lambda_{w,j} \cap \Lambda_{w,k}$. We construct similarly a function γ_k^m interverting j and k, we set $\gamma_{j,k}^m = \gamma_j^m \gamma_k^m$ and finally:

$$\gamma_m = \prod_{1\leqslant j<k\leqslant K_w^T} \gamma_{j,k}^m \qquad \left(\text{or} \prod_{0\leqslant j<k\leqslant K_w^T} \gamma_{j,k}^m \text{ if } w \in N_E\Gamma\right) \tag{4.8}$$

so that the sets $\Lambda_{w,k}\cap\mathcal{V}_{w,T}$ for $1\leqslant k\leqslant K_w^T$ (or $0\leqslant k\leqslant K_w^T$) do not intersect on the support of γ_m and:

$$\operatorname{mes}_\Lambda\left(\operatorname{supp}(1-\gamma_m)\cap\left(\cup_{j=0}^{K_w^T}\Lambda_{w,k}\right)\right)\leqslant\frac{1}{m} \tag{4.9}$$

For all $k\in[\![0,K_w^T]\!]$, the support of the function $x\mapsto(1-\gamma_m)(x,\partial\psi_k(x))$ is of measure less than $\frac{C}{m}$ in $\mathbb{R}^n$ where C only depends on Γ. $Op_h^w(\gamma_m)B_{w,k}^T$ is a lagrangian distribution microlocally supported in $\Lambda_{w,k}\cap\operatorname{supp}(\gamma_m)$ with symbols uniformly bounded in h and k, so there is $c\geqslant 0$ such that for all $h\in]0,h_0]$:

$$\left|\tilde{u}_h^T-Op_h^w(\gamma_m)\tilde{u}_h^T\right|\leqslant\frac{c}{m} \tag{4.10}$$

Moreover, for $j\neq k\in[\![0,K_w]\!]$ the distributions $Op_h^w(q\gamma_m)B_{w,j}^T$ and $Op_h^w(\tilde{q}\gamma_m)B_{w,k}^T$ have disjoint microsupports, so we have:

$$\left\langle Op_h^w(q\gamma_m)B_{w,j}^T,Op_h^w(\tilde{q}\gamma_m)B_{w,k}^T\right\rangle=\underset{h\to0}{O}(h^\infty) \tag{4.11}$$

Taking $m\in\mathbb{N}$ large enough and using (4.7), (4.10) et (4.11), we obtain for all $h\in]0,h_0]$:

$$\left|\langle Op_h^w(q)u_h,u_h\rangle-\delta_w\left\langle Op_h^w(q\gamma_m)B_{w,0}^T,B_{w,0}^T\right\rangle-\sum_{k=1}^{K_w^T}\left\langle Op_h^w(q\gamma_m)B_{w,k}^T,B_{w,k}^T\right\rangle\right|\leqslant\frac{\varepsilon}{3} \tag{4.12}$$

5. Convergence for intermediate times.

Let $k\in[\![1,K_w^T]\!]$. We know that $B_{w,k}^T$ is a lagrangian distribution of phase $\psi_{w,k}$ and of principal symbol $b_{w,k}$, hence we have:

$$\left\langle Op_h^w(q)Op_h^w(\gamma_m)B_{w,k}^T,B_{k,w}^T\right\rangle=\int_{\mathbb{R}^n}q(x,\partial\psi_{w,k}(x))\gamma_m(x,\partial\psi_{w,k}(x))\left|b_{w,k}(x)\right|^2\,dx+\underset{h\to0}{o}(1)$$

If m is large enough and h_0 small enough, we have for all $h\in]0,h_0]$:

$$\left|\left\langle Op_h^w(q)Op_h^w(\gamma_m)B_{w,k}^T,B_{w,k}^T\right\rangle-\int_{\mathbb{R}^n}q(x,\partial\psi_{w,k}(x))\left|b_{w,k}(x)\right|^2\,dx\right|\leqslant\frac{\varepsilon}{3K_w^T} \tag{4.13}$$

6. Convergence for small times.

It only remains to consider the term $\delta_w\left\langle Op_h^w(q)Op_h^w(\gamma_m)B_{w,0}^T,B_{w,0}^T\right\rangle$. We assume that w belongs to $N_E\Gamma$.

Let $\tau_1\in]0,\tau_0]$ and $v\in C_0^\infty(\mathbb{R}^{2n},[0,1])$ such that $\operatorname{supp}v\subset\tilde{\Gamma}(\tau_1)$ and v is equal to 1 in a neighborhood of $\operatorname{supp}A$. By proposition 3.5, if $\tau_1>0$ is small enough we have:

$$\left\|vB_{w,0}^T\right\|_{L^2(\mathbb{R}^n)}\leqslant\frac{\varepsilon}{6} \tag{4.14}$$

On the other hand, since $(1-v)$ vanishes around $\operatorname{supp}A$, we can write $(1-v(x))B_{w,0}^T$ as a lagrangian distribution (see proposition 3.6):

$$\begin{aligned}&\left\langle Op_h^w(q)Op_h^w(1-v)Op_h^w(\gamma_m)B_{w,0}^T,B_{w,0}^T\right\rangle\\&\quad=\int_{\mathbb{R}^n}(q\gamma_m)(x,\partial\psi(x))(1-v(x))\left|b_0(x)\right|^2\,dx+\underset{h\to0}{o}(1)\end{aligned}$$

Thus, if τ_1 and h_0 are small enough, then for all $h\in]0,h_0]$:

$$\left|\left\langle Op_h^w(q)Op_h^w(1-v)Op_h^w(\gamma_m)B_{w,0}^T,B_{w,0}^T\right\rangle-\int q\,d\mu_{w,0}\right|\leqslant\frac{\varepsilon}{6} \tag{4.15}$$

7. Conclusion. According to (4.12), (4.13) and (4.15), we can conclude that (4.6) holds. $\square$

5 Convergence toward a semiclassical measure

5.1 Large times control

For $R>0$, $d>0$ and $\sigma\in]-1,1[$ we note:

$$\begin{aligned}\Gamma_\pm(R,d,\sigma) &= \big\{(x,\xi)\in\mathbb{R}^{2n} : |x|\geqslant R, |\xi|\geqslant d \text{ and } \langle x,\xi\rangle \gtrless \sigma\,|x|\,|\xi|\big\}\\ \Gamma_\pm(d,\sigma) &= \big\{(x,\xi)\in\mathbb{R}^{2n} : |\xi|\geqslant d \text{ and } \langle x,\xi\rangle \gtrless \sigma\,|x|\,|\xi|\big\}\end{aligned}$$

As mentionned in the introduction, the following proposition states that the outgoing solution u_h is microlocally zero in the incoming region. The proof of this proposition is postponed to section 6.

Proposition 5.1. *Let $d>0$, $\sigma\in]0,1[$ and E_h such that $\operatorname{Im} E_h>0$ or E_h is positive and satisfies (1.4). Then there exists $R>0$ such that if $\omega_-,\omega\in\mathcal{S}_0$ are supported in $\Gamma_-(R,d,-\sigma)$ (respectively outside $\Gamma_-(R_1,d_1,-\sigma_1)$ for some $R_1<R$, $d_1<d$ and $\sigma_1<\sigma$) then:*

$$\left\|Op_h(\omega_-)(H_h-(E_0+i0))^{-1}Op_h(\omega)\right\| = \underset{h\to 0}{O}(h^\infty)$$

We now use this proposition to show that for T large enough, $\big\langle Op_h^w(q)u_h^T,u_h^T\big\rangle$ is a good approximation of $\langle Op_h^w(q)u_h,u_h\rangle$.

Proposition 5.2. *Let $q\in C_0^\infty(\mathbb{R}^{2n})$ be supported in $p^{-1}(I)$ and $\varepsilon>0$. Then there exists $T_0\geqslant 0$ such that for all $T\geqslant T_0$ we can find $h_T>0$ which satisfies:*

$$\forall h\in]0,h_T],\quad \left|\langle Op_h^w(q)u_h,u_h\rangle - \big\langle Op_h^w(q)u_h^T,u_h^T\big\rangle\right|\leqslant\varepsilon$$

Proof. **1.** Let $R_b\geqslant 0$ such that $\Gamma\subset B_{\mathbb{R}^n}(R_b)$, $\operatorname{supp} q\subset B_x(R_b)=\big\{(x,\xi)\in\mathbb{R}^{2n}:|x|<R_b\big\}$ and any trajectory of energy in J which leaves $B_x(R_b)$ never comes back (and goes to infinity). Let $\chi\in C_0^\infty(\mathbb{R}^n)$ supported in $B(2R_b)$ and equal to 1 on $B(R_b)$. Let $Q\in C_0^\infty(\mathbb{R}^{2n})$ supported in $p^{-1}(J)$ and equal to 1 in a neighborhood of $p^{-1}(I)\cap B_x(2R_b)$ and of $\operatorname{supp} q$. Let $T\geqslant 0$ and ω_- equal to 1 in the incoming region $\Gamma_-(R_b,-1/2)$ and zero outside $\Gamma_-(R_b/2,-1/4)$. We have:

$$\begin{aligned}Op_h^w(Q)u_h &= \frac{i}{h}\int_{t=0}^T Op_h^w(Q)U_h^E(t)S_h\,dt + Op_h^w(Q)U_h^E(T)u_h\\ &= Op_h^w(Q)u_h^T + Op_h^w(Q)U_h^E(T)Op_h^w(Q)u_h\\ &\quad + Op_h^w(Q)U_h^E(T)Op_h^w(1-Q)\chi(x)u_h\\ &\quad + Op_h^w(Q)U_h^E(T)Op_h^w(1-Q)(1-\chi(x))Op_h(\omega_-)u_h\\ &\quad + Op_h^w(Q)U_h^E(T)Op_h^w(1-Q)(1-\chi(x))Op_h(1-\omega_-)u_h\end{aligned}\tag{5.1}$$

For T large enough the last three terms are $O_{h\to 0}(\sqrt{h})$ respectively by the localization close to the E_0-energy hypersurface (proposition 2.10, which implies that $Op_h^w(1-Q)\chi(x)u_h$ is small), estimates on the incoming region ($Op_h^w(\omega_-)u_h$ is small by proposition 5.1, changing quantization is harmless here) and Egorov theorem ($Op_h^w(Q)U_h^E(T)Op_h(1-\omega_-)(1-\chi(x))$ is small). Hence we have:

$$\left(1-Op_h^w(Q)U_h^E(T)Op_h^w(\tilde Q)\right)Op_h^w(Q)u_h = Op_h^w(Q)u_h^T + \underset{h\to 0}{O}(\sqrt{h})\tag{5.2}$$

where $\tilde Q\in C_0^\infty(\mathbb{R}^{2n})$ is supported in $p^{-1}(J)$ and equal to 1 on the support of Q. Furthermore:

$$\left\|Op_h^w(Q)u_h^T\right\|^2 = \big\langle Op_h^w(Q)^2u_h^T,u_h^T\big\rangle \xrightarrow[h\to 0]{} \int Q^2 d\mu_T < +\infty$$

Hence for any (large enough) fixed T, the right-hand side of (5.2) is uniformly bounded in h. Moreover, by proposition 2.3, there exists T_0 such that for all $T \geqslant T_0$ there is $h_T > 0$ which satisfies:

$$\forall h \in]0, h_T], \quad \left\| Op_h^w(Q) U_h^E(T) Op_h^w(\tilde{Q}) \right\| \leqslant \frac{1}{2}$$

As a consequence, the operator $(1 - Op_h^w(Q)U_h^E(T)Op_h^w(\tilde{Q}))$ is invertible and its inverse is bounded uniformly in $T \geqslant T_0$ and $h \in]0, h_T]$. This proves that the quantity:

$$Op_h^w(Q)u_h = \left(1 - Op_h^w(Q)U_h^E(T)Op_h^w(\tilde{Q})\right)^{-1} Op_h^w(Q)u_h^T + \underset{h\to 0}{O}(\sqrt{h})$$

is bounded uniformly in $h \in]0, h_T]$ for fixed $T \geqslant T_0$ and hence is bounded uniformly for h small enough since the left hand side does not depend on T.

2. As for (5.1) we see that:

$$\begin{aligned} Op_h^w(q)u_h = {} & Op_h^w(q)u_h^T + Op_h^w(q)U_h^E(T)Op_h^w(Q)u_h \\ & + Op_h^w(q)U_h^E(T)Op_h^w(1-Q)\chi(x)u_h \\ & + Op_h^w(q)U_h^E(T)Op_h^w(1-Q)(1-\chi(x))Op_h(\omega_-)u_h \\ & + Op_h^w(q)U_h^E(T)Op_h^w(1-Q)(1-\chi(x))Op_h(1-\omega_-)u_h \end{aligned} \tag{5.3}$$

And as for (5.1) the last three terms are $\underset{h\to 0}{O}(\sqrt{h})$ by localization close to E_0-energy hypersurface, estimates in the incoming region and Egorov theorem. Moreover the second term is:

$$Op_h^w(q)U_h^E(T)Op_h^w(Q)u_h = Op_h^w(q)U_h^E(T)Op_h^w(\tilde{Q})\left(Op_h^w(Q)u_h\right) + \underset{h\to 0}{O}(\sqrt{h})$$

But $Op_h^w(Q)u_h$ is bounded uniformly in h and the operator $Op_h^w(q)U_h(T)Op_h(\tilde{Q})$ is of norm less than any $\delta > 0$ for T big enough and h small enough (depending of the chosen T). Hence we have proved:

$$\forall \delta > 0, \exists T_0 \geqslant 0, \forall T \geqslant T_0, \exists h_T > 0, \forall h \in]0, h_T], \quad \left\| Op_h^w(q)(u_h - u_h^T) \right\| \leqslant \delta \tag{5.4}$$

and in particular:

$$\exists C \geqslant 0, \forall T \geqslant T_0, \forall h \in]0, h_T], \quad \left\| Op_h^w(q)u_h^T \right\| \leqslant C \tag{5.5}$$

We consider $\tilde{q} \in C_0^\infty(\mathbb{R}^{2n})$ supported in $p^{-1}(I)$, equal to 1 on $\operatorname{supp} q$ and such that $Q = 1$ on a neighborhood of $\operatorname{supp} \tilde{q}$. We can assume that (5.4)-(5.5) hold for q and $\tilde{q}$. Let $\delta \in \left]0, \frac{\varepsilon}{4C}\right]$ and then T and h_T given by (5.4). For all $h \in]0, h_T]$ we have:

$$\begin{aligned} & \left| \langle Op_h^w(q)u_h, u_h \rangle - \langle Op_h^w(q)u_h^T, u_h^T \rangle \right| \\ & \quad = \left| \langle Op_h^w(q)u_h, Op_h^w(\tilde{q})u_h \rangle - \langle Op_h^w(q)u_h^T, Op_h^w(\tilde{q})u_h^T \rangle \right| + \underset{h\to 0}{O}(h^\infty) \\ & \quad \leqslant \left| \langle Op_h^w(q)(u_h - u_h^T), Op_h^w(\tilde{q})u_h^T \rangle \right| + \left| \langle Op_h^w(q)u_h, Op_h^w(\tilde{q})(u_h - u_h^T) \rangle \right| + \underset{h\to 0}{O}(h^\infty) \\ & \quad \leqslant \delta \left(\left\| Op_h^w(q)u_h^T \right\| + \left\| Op_h^w(\tilde{q})u_h^T \right\| \right) + \underset{h\to 0}{O}(h^\infty) \\ & \quad \leqslant \frac{\varepsilon}{2} + \underset{h\to 0}{O}(\sqrt{h}) \end{aligned}$$

and this last quantity is less than ε if we choose h small enough. □

5.2 Convergence of the partial semiclassical measure

Proposition 5.3. *There exists a Radon measure μ on $\mathbb{R}^{2n}$ such that for all $q \in C_0^\infty(\mathbb{R}^{2n})$:*

$$\int q \, d\mu_T \xrightarrow[T\to+\infty]{} \int q \, d\mu$$

and we have:

$$\langle Op_h^w(q)u_h, u_h\rangle \xrightarrow[h\to 0]{} \int q\, d\mu$$

Proof. **1.** We can assume that for any $w \in \mathbb{R}^{2n}$, the family of neighborhoods $\mathcal{V}_{w,T}, T \geqslant 0$, decreases when T increases. Let $T_1 \leqslant T_2 \in \mathbb{R}_+$. For $w \in \mathbb{R}^{2n}$ and $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $\mathcal{V}_{w,T_2} \subset \mathcal{V}_{w,T_1}$ we have:

$$\int q\, d\mu_{T_1} = \int q\, d\mu_{w,T_1} = \sum_{k=0}^{K_w^{T_1}} \int_q d\mu_{w,T_1,k} \leqslant \sum_{k=0}^{K_w^{T_2}} \int q\, d\mu_{w,T_2,k} = \int q\, d\mu_{T_2}$$

Since any $q \in C_0^\infty(\mathbb{R}^{2n})$ can be written as a finite sum $\sum q_i$ where q_i is supported in $\mathcal{V}_{w_i,T_2}$ for some w_i, the same applies for all $q \in C_0^\infty(\mathbb{R}^{2n})$. This proves that $\int q\, d\mu_T$ grows with T, and hence has a limit in $\mathbb{R}_+ \cup \{+\infty\}$ when T goes to $+\infty$.

2. If $\operatorname{supp} q \cap p^{-1}(\{E_0\}) = \emptyset$, then

$$\int q\, d\mu_T = 0 \xrightarrow[T\to+\infty]{} 0$$

This is consistent with corollary 2.11.

3. Now let $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in $p^{-1}(I)$, $\tilde{q}$ and C as in the proof of proposition 5.2 (see (5.5)). We have:

$$\int q\, d\mu_T = \lim_{h\to 0} \left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle = \lim_{h\to 0} \left\langle Op_h^w(q)u_h^T, Op_h^w(\tilde{q})u_h^T\right\rangle \leqslant C^2$$

As a result, $\int q\, d\mu_T$ as a finite limit when T goes to $+\infty$. This limit defines a nonnegative (each μ_T is a nonnegative measure) linear form on $C_0^\infty(\mathbb{R}^{2n})$. Let K be compact in $\mathbb{R}^{2n}$ and $Q \in C_0^\infty(\mathbb{R}^{2n})$ equal to 1 on K. Then for all $q \in C_0^\infty(\mathbb{R}^{2n})$ supported in K we have:

$$\left|\int q\, d\mu\right| \leqslant \lim_{T\to\infty} \int |q|\; d\mu_T \leqslant \|q\|_\infty \lim_{T\to\infty} \int Q\, d\mu_T \leqslant c\,\|q\|_\infty$$

and hence this limit is a continuous function of q (is the space of compactly supported continuous functions). Thus the application $q \mapsto \lim_{T\to+\infty} \int q\, d\mu_T$ can be extended to a nonnegative continuous linear form on the space of compactly supported continuous functions so, by Riesz theorem, there is a nonnegative Radon measure μ on $\mathbb{R}^{2n}$ such that:

$$\lim_{T\to\infty} \int q\, d\mu_T = \int q\, d\mu$$

4. For $q \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ there exists $T \geqslant 0$ such that:

$$0 \leqslant \int q\, d\mu - \int q\, d\mu_T \leqslant \frac{\varepsilon}{3}$$

According to proposition 5.2, if T is chosen large enough there is $h_T > 0$ such that:

$$\forall h \in]0, h_T], \quad \left|\langle Op_h^w(q)u_h, u_h\rangle - \left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle\right| \leqslant \frac{\varepsilon}{3}$$

and by theorem 4.4, there is $h_0 \in]0, h_T]$ such that for all $h \in]0, h_0]$ we have:

$$\left|\left\langle Op_h^w(q)u_h^T, u_h^T\right\rangle - \int q\, d\mu_T\right| \leqslant \frac{\varepsilon}{3}$$

Hence we get:

$$\forall h \in]0, h_0], \quad \left|\langle Op_h^w(q)u_h, u_h\rangle - \int q\, d\mu\right| \leqslant \varepsilon$$

which proves the proposition. □

5.3 Characterization of the semiclassical measure

We now finish the proof of theorem 1.1:

Proof. **1.** Statement (i) is already proved and similarly, (ii) is a consequence of the estimate in the incoming region (see proposition 5.1).

2. Let $q \in C_0^\infty(\mathbb{R}^{2n})$ such that $\operatorname{supp} q \cap (N_E\Gamma \cup \Lambda) = \emptyset$. We have:

$$\begin{aligned}\int q\,(H_p + 2\operatorname{Im} E_1 + 2V_2)d\mu &= \int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu \\ &= \lim_{T\to\infty} \int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu_T \\ &= 0\end{aligned}$$

according to (4.4) since the support of $(-H_p + 2\operatorname{Im} E_1 + 2V_2)q$ does not meet $N_E\Gamma \cup \Lambda$.

3. Let $w \in \Lambda$, $T \geqslant 0$ and $q \in C_0^\infty(\mathbb{R}^{2n})$ such that $\operatorname{supp} q \subset \mathcal{V}_{w,T}$.

Since $2ih\operatorname{Im} E_1 = E_h - \overline{E}_h + \underset{h\to 0}{o}(h)$ and $H_p(q) = \{p, q\}$ is the principal symbol of the operator $\frac{i}{h}[H_1^h, Op_h^w(q)]$, we have:

$$Op_h^w(H_p(q)) = \frac{i}{h}[H_1^h, Op_h^w(q)] + hOp_h^w(r_1) + \underset{h\to 0}{O}(h^2)$$

for some symbol $r_1 \in C_0^\infty(\mathbb{R}^{2n})$. But $\left\langle Op_h^w(r_1)B_{w,k}^T, B_{w,k}^T\right\rangle$ as a limit as h goes to 0 (which is $\int r_1 d\mu_{w,T,k}$, see step 5 in the proof of theorem 4.4) and $\left\|B_{w,k}^T\right\| = O(h^{-\frac{1}{2}})$, so:

$$\begin{aligned}&\int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu_{w,T,k} \qquad (5.6)\\ &= \lim_{h\to 0}\left\langle Op_h^w(-H_p(q) + 2\operatorname{Im} E_1 q + 2V_2 q)B_{w,k}^T, B_{w,k}^T\right\rangle \\ &= \lim_{h\to 0}\left\langle -\frac{i}{h}[H_1^h, Op_h^w(q)] + 2\operatorname{Im} E_1 Op_h^w(q) + 2V_2 Op_h^w(q)B_{w,k}^T, B_{w,k}^T\right\rangle \\ &= -\lim_{h\to 0}\frac{i}{h}\left\langle ((H_h - E_h)^* Op_h^w(q) - Op_h^w(q)(H_h - E_h))B_{w,k}^T, B_{w,k}^T\right\rangle \\ &= -\lim_{h\to 0}\frac{i}{h}\left(\left\langle Op_h^w(q)B_{w,k}^T, (H_h - E_h)B_{w,k}^T\right\rangle - \left\langle (H_h - E_h)B_{w,k}^T, Op_h^w(q)B_{w,k}^T\right\rangle\right) \\ &= 0\end{aligned}$$

according to proposition 4.3.

4. Let $q \in C_0^\infty(\mathbb{R}^{2n})$ and $\varepsilon > 0$. There exists $T \geqslant 0$ such that:

$$\int q\,d\mu_T \geqslant \int q\,d\mu - \frac{\varepsilon}{2}$$

We can find a finite number of $w_i \in \mathbb{R}^{2n}$ such that $\operatorname{supp} q \subset \cup\mathcal{V}_{w_i,T}$ and either $w_i \in N_E\Gamma\cup\Lambda$ or $\mathcal{V}_{w_i,T}\cap(N_E\Gamma\cup\Lambda) = \emptyset$. With a partition of unity, we can write $q = \sum q_i$ with $\operatorname{supp} q_i \subset \mathcal{V}_{w_i,T}$ and show the result for each q_i. So without loss of generality we can assume that $\operatorname{supp} q \subset \mathcal{V}_{w,T}$ for some $w \in N_E\Gamma \cup \Lambda$. According to (5.6) we have:

$$\begin{aligned}\int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu_T &= \sum_{j=0}^{K_w^T} \int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu_{w,T,k} \\ &= \int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q\,d\mu_{w,T,0}\end{aligned}$$

This is zero unless $w \in N_E\Gamma$, which we now assume. Let $g \in C_0^\infty(\mathbb{R})$ supported in $]-\infty, 1]$ with $g = 1$ near 0. For $m \in \mathbb{N}$ and $(x,\xi) \in \tilde{\Gamma}(\tau_0) \times \mathbb{R}^n$ we set $g_m(x,\xi) = g(mt_x)$. In particular the function $(1 - g_m)q$ vanishes near $N_E\Gamma$, so:

$$\int (-H_p + 2V_2 + 2\operatorname{Im} E_1)(1 - g_m)q \, d\mu = 0$$

Then since g_m is supported in $\tilde{\Gamma}(0,\tau_0)$ for all $m \in \mathbb{N}$, we can use (2.4) to have:

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} (-H_p + 2\operatorname{Im} E_1 + 2V_2)q \, d\mu_{w,T,0} \\
&\quad = \int_{\mathbb{R}^{2n}} (-H_p + 2\operatorname{Im} E_1 + 2V_2)qg_m \, d\mu_{w,T,0} \\
&\quad = \int_{\tilde{\Gamma}(0,\tau_0)} (-H_p + 2\operatorname{Im} E_1 + 2V_2)(qg_m)(x, \partial\psi(x)) \, |b_0(x)|^2 \, dx \\
&\quad = 2^{n-d} \int_0^{\tau_0}\!\!\int_{N_E\Gamma} t^{n-d-1} \, |\xi| \left(1 + \underset{t\to 0}{O}(t)\right) |b_0(\overline{x}(t,z,\xi))|^2 \\
&\qquad\qquad \times (-H_p + 2\operatorname{Im} E_1 + 2V_2)(qg_m)(\overline{x}(t,z,\xi), \partial\psi(\overline{x}(t,z,\xi))) \, d\tilde{\sigma}(z,\xi) \, dt
\end{aligned}$$

According to (3.7) we have $(x, \partial\psi(x)) = \phi^{t_x}(z_x, \xi_x)$. On the other hand, by (3.25) and (3.27) we have:

$$2^{n-d} t^{n-d-1} \, |\xi| \, |b_0(\overline{x}(t,z,\xi))|^2 \xrightarrow[t\to 0]{} \pi (2\pi)^{d-n} A(z)^2 \, |\xi|^{-1} \, \hat{S}(\xi)^2 =: c(z,\xi) \tag{5.7}$$

so:

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} (-H_p + 2\operatorname{Im} E_1 + 2V_2)q \, d\mu_{w,T,0} \\
&\quad = -\int_0^{\tau_0}\!\!\int_{N_E\Gamma} (\partial_t - 2\operatorname{Im} E_1 - 2V_2)(q(\phi^t(z,\xi))g(mt))c(z,\xi)\left(1 + \underset{t\to 0}{O}(t)\right) d\tilde{\sigma}(z,\xi) \, dt \\
&\quad = -\int_0^{\tau_0}\!\!\int_{N_E\Gamma} g(tm)(\partial_t - 2\operatorname{Im} E_1 - 2V_2)(q(\phi^t(z,\xi)))c(z,\xi)\left(1 + \underset{t\to 0}{O}(t)\right) d\tilde{\sigma}(z,\xi) \, dt \\
&\qquad - \int_0^{\tau_0}\!\!\int_{N_E\Gamma} mg'(tm)q(\phi^t(z,\xi))c(z,\xi)\left(1 + \underset{t\to 0}{O}(t)\right) d\tilde{\sigma}(z,\xi) \, dt
\end{aligned}$$

and hence:

$$\begin{aligned}
&\left| \int (-H_p + 2\operatorname{Im} E_1 + 2V_2)q \, d\mu_{w,T,0} - \int_{N_E\Gamma} q(z,\xi)c(z,\xi) \, d\tilde{\sigma}(z,\xi) \right| \\
&\quad \leqslant O\left(\frac{1}{m}\right) + \left| \int_0^{\tau_0}\!\!\int_{N_E\Gamma} mg'(tm)\big(q(z,\xi) - q(\phi^t(z,\xi))\big)c(z,\xi) \, d\tilde{\sigma}(z,\xi) \, dt \right| \\
&\quad \leqslant O\left(\frac{1}{m}\right) + \int_0^{\tau_0}\!\!\int_{N_E\Gamma} m \, |g'(tm)| \sup_{0 \leqslant t \leqslant \frac{1}{m}} \big| q(z,\xi) - q(\phi^t(z,\xi)) \big| \, c(z,\xi) \, d\tilde{\sigma}(z,\xi) \, dt \\
&\quad = O\left(\frac{1}{m}\right)
\end{aligned}$$

It only remains to choose m so large that the rest is less than $\frac{\varepsilon}{2}$. □

As said in the introduction, μ is actually characterized by the three properties of theorem 1.1 and is given by (1.10):

Proposition 5.4. *Let ν be a Radon measure on $\mathbb{R}^{2n}$ which satisfies the three properties of theorem 1.1. Then for all $q \in C_0^\infty(\mathbb{R}^{2n})$ we have:*

$$\int_{\mathbb{R}^{2n}} q \, d\nu = \int_0^{+\infty} \int_{N_E\Gamma} c(z,\xi) q(\phi^t(z,\xi)) e^{-2t\operatorname{Im} E_1 - 2\int_0^t V_2(\overline{x}(s,z,\xi)) \, ds} \, d\tilde{\sigma}(z,\xi) \, dt \tag{5.8}$$

where the function c is defined in (5.7).

Proof. Let I_1 be an open interval such that $\overline{I} \subset I_1 \subset \overline{I_1} \subset J$. Let $q \in C_0^\infty(\mathbb{R}^{2n})$. According to property (i), if $\operatorname{supp} q \subset p^{-1}(\mathbb{R} \setminus I)$ then $\int q\, d\nu = 0$ which is consistent with (5.8), since both sides are zero. So we can assume that $\operatorname{supp} q \subset p^{-1}(I_1)$.

Using property (iii) we see that:

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{2n}} (q\circ\phi^t) e^{-2t\operatorname{Im}E_1 - 2\int_0^t V_2\circ\phi^{t-s}\,ds}\, d\nu \\
&\quad = \int_{\mathbb{R}^{2n}} (H_p - 2\operatorname{Im}E_1 - 2V_2)\left((q\circ\phi^t) e^{-2t\operatorname{Im}E_1 - 2\int_0^t V_2\circ\phi^{t-s}\,ds}\right) d\nu \\
&\quad = -\int_{N_E\Gamma} c(z,\xi)\left((q\circ\phi^t) e^{-2t\operatorname{Im}E_1 - 2\int_0^t V_2\circ\phi^{t-s}\,ds}\right)(z,\xi)\, d\tilde{\sigma}(z,\xi)
\end{aligned}$$

and hence, for all $\tau \geqslant 0$:

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} q\, d\mu = &\int_{\mathbb{R}^{2n}} (q\circ\phi^\tau) e^{-2\tau\operatorname{Im}E_1 - 2\int_0^\tau V_2\circ\phi^{\tau-s}\,ds}\, d\nu \\
&+ \int_0^\tau \int_{N_E\Gamma} c(z,\xi)\left((q\circ\phi^t) e^{-2t\operatorname{Im}E_1 - 2\int_0^t V_2\circ\phi^{t-s}\,ds}\right)(z,\xi)\, d\tilde{\sigma}(z,\xi)\, dt
\end{aligned}$$

So we only have to prove that:

$$\int_{\mathbb{R}^{2n}} (q\circ\phi^\tau) e^{-2\tau\operatorname{Im}E_1 - 2\int_0^\tau V_2\circ\phi^{\tau-s}\,ds}\, d\nu \xrightarrow[\tau\to+\infty]{} 0$$

For $R \geqslant 0$ we set: $K_R = p^{-1}(\overline{I}_1) \cap B_x(R)$. According to property (ii), we can find $R \geqslant 0$ such that ν vanishes on $\Gamma_-(R, -\frac{1}{2})$ and:

$$\bigcup_{t\geqslant 0} \operatorname{supp}(q\circ\phi^t) \subset \Gamma_-\left(R, -\frac{1}{2}\right) \cup K_R$$

Let $\chi \in C_0^\infty(\mathbb{R}^{2n})$ supported in $p^{-1}(J)$ and equal to 1 on K_R. For $\tau \geqslant 0$, since ν vanishes on $\Gamma_-\left(R, -\frac{1}{2}\right)$:

$$\int_{\mathbb{R}^{2n}} (q\circ\phi^\tau) e^{-2t\operatorname{Im}E_1 - 2\int_0^\tau V_2\circ\phi^{\tau-s}\,ds}\, d\nu = \int_{\mathbb{R}^{2n}} \chi(q\circ\phi^\tau) e^{-2t\operatorname{Im}E_1 - 2\int_0^\tau V_2\circ\phi^{\tau-s}\,ds}\, d\nu$$

As ν is a Radon measure, there is a constant $C \geqslant 0$ such that for all $\tilde{q} \in C_0^\infty(\mathbb{R}^{2n})$ with $\operatorname{supp} q \subset \operatorname{supp}\chi$ we have:

$$\left|\int_{\mathbb{R}^{2n}} \tilde{q}\, d\nu\right| \leqslant C\left\|\tilde{q}\right\|_{L^\infty(\mathbb{R}^{2n})}$$

so we only need to prove that:

$$\sup_{w\in\mathbb{R}^{2n}} \left|\chi(w)(q\circ\phi^\tau)(w) e^{-2\tau\operatorname{Im}E_1 - 2\int_0^\tau (V_2\circ\phi^{\tau-s})(w)\,ds}\right| \xrightarrow[\tau\to+\infty]{} 0$$

This is clear if $\operatorname{Im}E_1 > 0$. Otherwise, this can be done with lemma 2.2 as in the proof of proposition 2.3. $\square$

6 Estimate of the outgoing solution in the incoming region

The theorem we want to prove in this section is the following:

Theorem 6.1. *Let $N \in \mathbb{N}$ and $E_h = E_0 + O(h)$ be an energy such that for all $h \in]0, h_0]$, $\operatorname{Im} E_h > 0$ or E_h satisfies (1.4). Let $d > 0$ and $\sigma \in]0,1[$. Then there exits $\nu \in \mathbb{N}$ and $R > 0$ such that if the symbols $\omega_+, \omega \in \mathcal{S}_0$ have supports in $\Gamma_+(R, d, \sigma)$ (respectively outside $\Gamma_+(R_1, d_1, \sigma_1)$ with $R_1 < R$, $d_1 < d$ and $\sigma_1 < \sigma$) then for all $\alpha > \frac{1}{2}$ we have:*

$$\left\| \langle x \rangle^{-\alpha} Op_h(\omega)(H_h - (E_h + i0))^{-1} Op_h(\omega_+) \langle x \rangle^{-\nu} \right\| = \underset{h \to 0}{O}(h^N) \tag{6.1}$$

Similarly, if $\operatorname{supp} \omega_- \subset \Gamma_-(R, d, -\sigma)$ *and* $\operatorname{supp} \omega \cap \Gamma_-(R_1, d_1, -\sigma_1) = \emptyset$ *then:*

$$\left\| \langle x \rangle^{-\alpha} Op_h(\omega)(H_h^* - (E_h - i0))^{-1} Op_h(\omega_-) \langle x \rangle^{-\nu} \right\| = \underset{h \to 0}{O}(h^N) \tag{6.2}$$

Remark. This is the analog of lemma 2.3 in [RT89] in the dissipative case. Note that here ν is different from α and may be large.

Remark. Taking the adjoint in (6.2) gives:

$$\left\| \langle x \rangle^{-\nu} Op_h(\omega_-)(H_h - (E_h + i0))^{-1} Op_h(\omega) \langle x \rangle^{-\alpha} \right\| = \underset{h \to 0}{O}(h^N)$$

which proves proposition 5.1. This theorem proves that the solution $u_h = (H_h - (E + i0))^{-1} S_h$ is microlocally zero in the incoming region.

To prove this theorem we follow [Wan88]. In particular we use the following result taken from [IK85]:

Proposition 6.2. *Let $d_0 \in]0, d_1[$ and $\sigma_0 \in]0, \sigma_1[$. There exists $R_0 > 0$ and $\phi_\pm \in C^\infty(\mathbb{R}^{2n})$ satisfying:*

$$\forall (x, \xi) \in \Gamma_\pm(R_0, d_0, \pm\sigma_0), \quad |\nabla_x \phi_\pm(x, \xi)|^2 + V_1(x) = |\xi|^2 \tag{6.3}$$

and:

$$\forall (x, \xi) \in \mathbb{R}^{2n}, \forall \alpha, \beta \in \mathbb{N}^n, \qquad \left| \partial_x^\alpha \partial_\xi^\beta (\phi_\pm(x, \xi) - \langle x, \xi \rangle) \right| \leqslant C_{\alpha,\beta} \langle x \rangle^{1 - \rho - |\alpha|} \tag{6.4}$$

for some $\rho > 0$.

Without loss of generality we may assume that this is the same constant ρ as in (1.3).

Remark. As mentioned in [Wan88] (see (2.4)), we can assume that the constants $C_{\alpha,\beta}$ in (6.4) are as small as we wish if we take R large enough. Indeed, if we take a function $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi(x) = 0$ if $|x| \leqslant \frac{1}{2}$ and $\chi(x) = 1$ if $|x| \geqslant 1$, and, for $R > R_0$:

$$\phi_{R,\pm} : (x, \xi) \mapsto (\phi_\pm(x, \xi) - \langle x, \xi \rangle) \chi\left(\frac{x}{R}\right) + \langle x, \xi \rangle \tag{6.5}$$

Then:

$$\forall (x, \xi) \in \Gamma_\pm(R, d_0, \sigma_0), \quad |\nabla_x \phi_{R,\pm}(x, \xi)|^2 + V_1(x) = |\xi|^2 \tag{6.6}$$

and for any $\rho_1, \rho_2 > 0$ such that $\rho = \rho_1 + \rho_2$:

$$\forall (x, \xi) \in \mathbb{R}^{2n}, \quad \left| \partial_x^\alpha \partial_\xi^\beta (\phi_{R,\pm}(x, \xi) - \langle x, \xi \rangle) \right| \leqslant C_{\alpha,\beta} R^{-\rho_1} \langle x \rangle^{1 - \rho_2 - |\alpha|} \tag{6.7}$$

where $C_{\alpha,\beta}$ does not depend on R.

We are going to use the Fourier integral operators $I_h(a, \phi)$ defined as follows:

$$I_h(a, \phi) u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(\phi(x,\xi) - \langle y, \xi \rangle)} a(x, \xi) u(y) \, dy \, d\xi$$

As in [Wan88], the idea of the proof is to find two symbols a and e such that:

$$U_h(t) I_h(a, \phi) \approx I_h(a, \phi) U_0^h(t) \quad \text{and} \quad Op_h(\omega_+) \approx I_h(a, \phi) I_h(e, \phi)^*$$

when h goes to 0. For a short range absorption coefficient V_2, we can actually do as in [Wan88], but in the long range case, we have to consider a time dependant symbol $a(t,h)$. In this situation we have:

$$\begin{aligned}&U_h(t)I_h(a(t,h),\phi_\pm) - I_h(a(t,h),\phi_\pm)U_0^h(t) \\ &\quad = \int_0^t U_h(t)\left(-\frac{i}{h}H_hI_h(a(s,h),\phi_\pm) + I_h(\partial_t a(s,h),\phi_\pm) + \frac{i}{h}I_h(a(s,h),\phi_\pm)H_0^h\right)U_0^h(t-s)\,ds\end{aligned} \tag{6.8}$$

Proposition 6.3. *Let $a(t,h) \in \mathcal{S}_b$ be a time-dependant symbol, $\phi = \phi_+$ or ϕ_- given by proposition 6.2 and $h \in]0,1]$. Then we have:*

$$-\frac{i}{h}H_hI_h(a(t,h),\phi) + I_h(\partial_t a(t,h),\phi) + \frac{i}{h}I_h(a(t,h),\phi)H_0^h = I_h(p(t,h),\phi)$$

where:

$$\begin{aligned}&p(t,h) \\ &\quad = -\frac{i}{h}(|\partial_x\phi|^2 + V_1 - \xi^2)a(t,h) + \Big(\partial_t a(t,h) - 2\partial_x a(t,h).\partial_x\phi - a(t,h)\Delta_x\phi - a(t,h)V_2\Big) \\ &\quad\quad + ih\Delta_x a(t,h)\end{aligned} \tag{6.9}$$

Remark. If moreover $a(t,h)$ is of the form:

$$a(t,h) = \sum_{j=0}^{N} h^j a_j(t)$$

with $a_j \in \mathcal{S}_b$ for all $j \in [\![0,N]\!]$, then $p(t,h)$ takes the form:

$$\begin{aligned}p(t,h) = &-\frac{i}{h}(|\partial_x\phi|^2 + V_1 - \xi^2)a(t,h) + \Big(\partial_t a_0(t,h) - 2\partial_x a_0(t).\partial_x\phi - a_0(t)\Delta_x\phi - a_0(t)V_2\Big) \\ &+ \sum_{j=1}^{N} h^j\Big(\partial_t a_j(t,h) - 2\partial_x a_j(t).\partial_x\phi - a_j(t)\Delta_x\phi - a_j(t)V_2 + i\Delta_x a_{j-1}(t)\Big) \\ &+ ih^{N+1}\Delta_x a_N(t)\end{aligned}$$

This gives the transport equations the symbols a_j have to satisfy if we want $I_h(p(t,h),\phi) = \underset{h\to 0}{O}(h^{N+1})$.

Remark. Similarly we have:

$$-\frac{i}{h}H_h^*I_h(a(t,h),\phi) + I_h(\partial_t a(t,h),\phi) + \frac{i}{h}I_h(a(t,h),\phi)H_0^h = I_h(p_*(t,h),\phi)$$

where:

$$\begin{aligned}&p_*(t,h) \\ &\quad = -\frac{i}{h}(|\partial_x\phi|^2 + V_1 - \xi^2)a(t,h) + \Big(\partial_t a(t,h) - 2\partial_x a(t,h).\partial_x\phi - a(t,h)\Delta_x\phi + a(t,h)V_2\Big) \\ &\quad\quad + ih\Delta_x a(t,h)\end{aligned}$$

Lemma 6.4. *Let ϕ be a function which satisfies (6.4). Then for all $(x,\xi) \in \mathbb{R}^{2n}$, the Cauchy problem:*

$$\begin{cases}\frac{\partial r}{\partial t}(t,x,\xi) = \partial_x\phi(r(t,x,\xi),\xi) \\ r(0,x,\xi) = x\end{cases}$$

has a unique solution defined on $\mathbb{R}$. Furthermore, for $\gamma \in]0,\sigma_1[$, if R is large enough, we have the following properties:

(i) For $(x,\xi) \in \Gamma_\pm(d_1, \pm\sigma_1)$ *and* $\pm t \geqslant 0$ *we have:*

$$|r(t,x,\xi)| \geqslant |x| + (\sigma_1 - \gamma) d_1 |t| \tag{6.10}$$

(ii) For $(x,\xi) \in \Gamma_\pm(d_1, \pm\sigma_1)$*,* $\pm t \geqslant 0$ *and* $|\alpha| + |\beta| \geqslant 1$*, there is a constant* $c_{\alpha,\beta}$ *such that:*

$$\left|\partial_x^\alpha \partial_\xi^\beta r(t,x,\xi)\right| \leqslant c_{\alpha,\beta} \max(|t|, \langle x \rangle) \langle x \rangle^{-|\alpha|} \tag{6.11}$$

Proof. Let $(x,\xi) \in \mathbb{R}^{2n}$. We have:

$$r(t,x,\xi) = x + t\xi + \int_0^t (\partial_x \phi(r(s,x,\xi),\xi) - \xi)\, ds \tag{6.12}$$

where $r(\cdot, x, \xi)$ is defined, that is everywhere since $(\partial_x \phi(r(t,x,\xi),\xi) - \xi)$ is bounded according to (6.4).

(i) By (6.7), if R is large enough we can assume that:

$$\forall (x,\xi) \in \mathbb{R}^{2n}, \quad |\partial_x \phi(x,\xi) - \xi| \leqslant \gamma d_1$$

and hence:

$$|r(t,x,\xi) - x - t\xi| \leqslant |t|\, \gamma d_1$$

If $(x,\xi) \in \Gamma_\pm(d_1, \pm\sigma_1)$ and $\pm t \geqslant 0$, then:

$$|x + t\xi| \geqslant \frac{1}{|x|} \langle x, x + t\xi \rangle \geqslant |x| + \sigma_1 |t| |\xi| \geqslant |x| + |t| \sigma_1 d_1$$

so:

$$|r(t,x,\xi)| \geqslant |x + t\xi| - \gamma |t| d_1 \geqslant |x| + (\sigma_1 - \gamma) d_1 |t|$$

which proves (6.10).

(ii) We prove (6.11) by induction on $|\alpha| + |\beta|$, beginning by the case $|\alpha| = 1$, $\beta = 0$. Let $\pm t \geqslant 0$ and $(x,\xi) \in \Gamma_+(d_1, \sigma_1)$. We have:

$$\partial_t \partial_x r(t,x,\xi) = \partial_x^2 \phi(r(t,x,\xi),\xi).\partial_x r(t,x,\xi)$$

According to Gronwall lemma, (6.4) and (6.10), we obtain the estimate:

$$\begin{aligned}
\|\partial_x r(t,x,\xi)\| &\leqslant \exp\left(\int_0^t \left\|\partial_x^2 \phi(r(s,x,\xi),\xi)\right\| \, ds\right) \leqslant \exp\left(\int_0^t c \langle r(s,x,\xi) \rangle^{-1-\rho} \, ds\right) \\
&\leqslant \exp\left(\int_0^t c \langle s \rangle^{-1-\rho} \, ds\right) \leqslant c \leqslant c \max(|t|, \langle x \rangle) \langle x \rangle^{-1}
\end{aligned}$$

Similarly, if $\alpha = 0$ and $|\beta| = 1$ we have:

$$\partial_t \partial_\xi r(t,x,\xi) = \partial_x^2 \phi(r(t,x,\xi),\xi).\partial_\xi r(t,x,\xi) + \partial_x \partial_\xi \phi(r(t,x,\xi),\xi)$$

and then:

$$\|\partial_t \partial_\xi r(t,x,\xi)\| \leqslant \left|\int_{s=0}^t \|\partial_x \partial_\xi \phi(r(s,x,\xi),\xi)\| \exp\left(\int_{\tau=s}^t \left\|\partial_x^2 \phi(r(\tau,x,\xi),\xi)\right\| \, d\tau\right) ds\right| \leqslant c\,|t|$$

We now assume that we have proved (6.11) for $1 \leqslant |\alpha| + |\beta| \leqslant k \in \mathbb{N}^*$ and we consider α and β such that $|\alpha| + |\beta| = k + 1$. For $j \in [\![1, n]\!]$ we have:

$$\begin{aligned}
\partial_t \partial_x^\alpha \partial_\xi^\beta r_j(t,x,\xi) &= \partial_x^\alpha \partial_\xi^\beta (\partial_{x_j} \phi(r(t,x,\xi),\xi)) \\
&= \sum_{l=1}^n \partial_{x_l,x_j}^2 \phi(r(t,x,\xi),\xi)\, \partial_x^\alpha \partial_\xi^\beta r_l(t,x,\xi) + B_j(t,x,\xi)
\end{aligned}$$

where B_j is a sum of terms of the form:

$$(\partial_x^\gamma \partial_\xi^\delta \partial_{x_j}\phi)(r(t,x,\xi),\xi)\prod_{s=1}^{|\gamma|}(\partial_x^{\alpha_s}\partial_\xi^{\beta_s} r_{l_s})(t,x,\xi)$$

with $|\gamma|+|\delta|\geqslant 2$ and for all $s: l_s \in [\![1,n]\!]$, $|\alpha_s|+|\beta_s|\leqslant k$, $\sum \alpha_s=\alpha$ and $\delta+\sum\beta_s=\beta$. Then B_j is smaller than:

$$\langle r(t,x,\xi)\rangle^{-|\gamma|-\rho}\prod_{s=1}^{|\gamma|}\max(|t|,\langle x\rangle)\langle x\rangle^{-|\alpha_s|}\leqslant c\langle x\rangle^{-\alpha}$$

and finally (6.11) holds since:

$$\left\|\partial_t\partial_x^\alpha\partial_\xi^\beta r(t,x,\xi)\right\|\leqslant\left|\int_{s=0}^t\|B(t,x,\xi)\|\exp\left(\int_{\tau=s}^t\left\|\partial_x^2\phi(r(\tau,x,\xi),\xi)\right\|d\tau\right)ds\right|\leqslant c|t|\langle x\rangle^{-\alpha}$$

□

Let $r_\pm$ be the functions defined in this proposition for $\phi=\phi_\pm$ and:

$$F_\pm(t,x,\xi)=\Delta_x\phi_\pm(r_\pm(t,x,\xi),\xi)\pm V_2(r_\pm(t,x,\xi))$$

In particular we have:

$$F_\pm(0,x,\xi)=\Delta_x\phi_\pm(x,\xi)\pm V_2(x)\quad\text{and}\quad F_\pm(t,r_\pm(s,x,\xi),\xi)=F_\pm(t+s,x,\xi)$$

Proposition 6.5. *The functions $a_{j,\pm}(t,h), j\in\mathbb{N}$ defined by:*

$$a_{0,\pm}(t,x,\xi)=\exp\left(-\int_{s=0}^t(F_\pm(2s,x,\xi))\,ds\right)$$

and for $j\geqslant 1$:

$$a_{j,\pm}(t,x,\xi)=i\int_{\tau=0}^t\Delta_x a_{j-1,\pm}(\tau,r_\pm(2\tau,x,\xi),\xi)a_0(\tau,x,\xi)\,d\tau$$

are solutions of the transport equations:

$$\partial_t a_{0,\pm}(t,h)-2\partial_x a_{0,\pm}(t).\partial_x\phi_\pm-a_{0,\pm}(t)\Delta_x\phi_\pm\mp a_{0,\pm}(t)V_2=0\tag{6.13}$$

and for $j\geqslant 1$:

$$\partial_t a_{j,\pm}(t,h)-2\partial_x a_{j,\pm}(t).\partial_x\phi_\pm-a_{j,\pm}(t)\Delta_x\phi_\pm\mp a_{j,\pm}(t)V_2+i\Delta_x a_{j-1}(t)=0\tag{6.14}$$

and satisfy estimates:

$$\text{for }\pm t\geqslant 0,(x,\xi)\in\Gamma_\pm(d_1,\pm\sigma_1),\quad\left|\partial_x^\alpha\partial_\xi^\beta a_{j,\pm}(t,x,\xi)\right|\leqslant c_{\alpha,\beta}|t|^{j+(|\alpha|+|\beta|)(1-\rho)}\langle x\rangle^{-|\alpha|}\tag{6.15}$$

Proof. We prove (6.15). For $\alpha,\beta\in\mathbb{N}^n$, the derivative $\partial_x^\alpha\partial_\xi^\beta a_{0,\pm}(t,x,\xi,h)$ is a sum of terms of the form:

$$\prod_{k=1}^J\partial_x^{\mu_k}\partial_\xi^{\nu_k}\left(\int_0^t F_\pm(2s,x,\xi)\,ds\right)a_{0,\pm}(t,x,\xi)$$

with $\sum\mu_k=\alpha$, $\sum\nu_k=\beta$ and for all $k\in[\![1,J]\!]$: $|\mu_k|+|\nu_k|\geqslant 1$ (and in particular $J\leqslant|\alpha|+|\beta|$). We first remark that according to (6.4) and (6.10) together with nonnegativeness of V_2 the symbol a_0 is bounded uniformly in $\pm t\geqslant 0$. Hence we have to prove:

$$\left|\int_0^t\partial_x^{\mu_k}\partial_\xi^{\nu_k}F_\pm(2s,x,\xi)\,ds\right|\leqslant c_{\alpha,\beta}|t|^{(|\mu_k|+|\nu_k|)(1-\rho)}\langle x\rangle^{-|\mu_k|}$$

Let $\pm t \geqslant 0$, $(x,\xi) \in \Gamma_\pm(d_1, \pm\sigma_1)$ and $\mu, \nu \in \mathbb{N}^n$. Then:

$$\partial_x^\mu \partial_\xi^\nu \left(\int_0^t F_\pm(2s, x, \xi)\, ds \right)$$

is a sum of terms of the form:

$$\int_0^t \partial_x^\delta \partial_\xi^\lambda (\Delta_x \phi_\pm + V_2)(r_\pm(2s,x,\xi),\xi) \prod_{k=1}^{|\delta|} \partial_x^{\mu_k} \partial_\xi^{\nu_k} r_\pm(2s,x,\xi)\, ds \tag{6.16}$$

with $\sum_{j=1}^{|\delta|} \mu_k = \mu$, $\sum_{j=1}^{|\delta|} \nu_k + \lambda = \nu$ and for all $k \in [\![1, |\delta|]\!]$: $|\mu_k| + |\nu_k| \geqslant 1$. By (1.3), (6.4) and (6.11) we have:

$$\left| \partial_x^\mu \partial_\xi^\nu \left(\int_0^t F_\pm(s,x,\xi)\, ds \right) \right| \leqslant c \left|t\right|^{1-\rho} \langle x \rangle^{-|\mu|}$$

this proves (6.15) for $j = 0$. We now prove the general case by induction. For $\alpha, \beta \in \mathbb{N}^n$ the derivative $\partial_x^\alpha \partial_\xi^\beta a_{j+1,\pm}(t,x,\xi)$ is a sum of terms of the form:

$$i \int_{\tau=0}^t \partial_x^\mu \partial_\xi^\nu (\Delta_x a_{j,\pm}(t, r_\pm(2(\tau - t), x, \xi), \xi)) \times \partial_x^{\alpha-\mu} \partial_\xi^{\beta-\nu} a_{0,\pm}(\tau, x, \xi)\, d\tau$$

We already know that for $\tau \in [0,t]$:

$$\left| \partial_x^{\alpha-\mu} \partial_\xi^{\beta-\nu} a_{0,\pm}(\tau,x,\xi) \right| \leqslant c \left|t\right|^{(1-\rho)(|\alpha-\mu|+|\beta-\nu|)} \langle x \rangle^{-|\alpha-\mu|}$$

So it remains to show:

$$\left| \partial_x^\mu \partial_\xi^\nu (\Delta_x a_{j,\pm}(\tau, r_\pm(2\tau,x,\xi),\xi)) \right| \leqslant c \left|t\right|^{j+(1-\rho)(|\mu|+|\nu|)} \langle x \rangle^{-|\mu|}$$

But $\partial_x^\mu \partial_\xi^\nu (\Delta_x a_{j,\pm}(\tau, r_\pm(2\tau,x,\xi),\xi))$ is a sum of terms of the form:

$$(\partial_x^\delta \partial_\xi^\lambda \Delta_x a_{j,\pm})(t, r_\pm(2\tau,x,\xi),\xi) \prod_{k=1}^{|\delta|} (\partial_x^{\mu_k} \partial_\xi^{\nu_k} r_\pm)(2\tau,x,\xi)$$

with $\mu = \sum_{k=1}^{|\delta|} \mu_k$ and $\nu = \lambda + \sum_{k=1}^{|\delta|} \nu_k$, and:

$$\begin{aligned}
&\left| (\partial_x^\delta \partial_\xi^\lambda \Delta_x a_{j,\pm})(\tau, r_\pm(2\tau,x,\xi),\xi) \prod_{j=1}^{|\delta|} (\partial_x^{\mu_j} \partial_\xi^{\nu_j} r_\pm)(2\tau,x,\xi) \right| \\
&\qquad \leqslant c \left|\tau\right|^{j+(1-\rho)(|\delta|+|\lambda|+2)} \langle r_\pm(2\tau,x,\xi) \rangle^{-|\delta|-2} \max(\left|2\tau\right|, \langle x \rangle)^\delta \langle x \rangle^{-\sum_{j=1}^{|\delta|} \mu_j} \\
&\qquad \leqslant c \left|t\right|^{j+(1-\rho)(|\delta|+|\lambda|)} \langle x \rangle^{-|\mu|}
\end{aligned}$$

which concludes the proof after integration over $\tau \in [0,t]$. □

Remark. This is for this part of the proof that we need a time-dependant symbol. Indeed, following exactly the proof of [Wan88] would have led to consider:

$$a_0(x,\xi) = \exp\left(\int_0^\infty F(t,x,\xi)\, dt \right)$$

which may have no sense for a long range imaginary part of the potential V_2. For a short range potential we do not have such a problem and the sign of V_2 we have used here does not matter.

Let σ_2 and σ_3 such that $\sigma_1 < \sigma_2 < \sigma_3 < \sigma$, R_2 and R_3 such that $R_1 < R_2 < R_3 < R$ and d_2, d_3 such that $d_1 < d_2 < d_3 < d$. We consider functions $\rho_1 \in C^\infty(\mathbb{R})$ such that $\rho_1(s) = 0$ if $s \leqslant \sigma_2$ and 1 if $s \geqslant \sigma_3$, $\rho_2 \in C^\infty(\mathbb{R})$ such that $\rho_2(s) = 0$ and $s \leqslant d_2$ and 1 if $s \geqslant d_3$ and $\rho_3 \in C^\infty(\mathbb{R})$ such that $\rho_3(s) = 0$ if $s \leqslant R_2$ and $\rho_3(s) = 1$ if $s \geqslant R_3$. Then we set:

$$b_\pm(t,x,\xi,h) = \psi_\pm(x,\xi)\sum_{j=0}^{N} h^j a_{j,\pm}(t,x,\xi) \quad \text{where: } \psi_\pm(x,\xi) = \rho_1\left(\frac{\pm\langle x,\xi\rangle}{|x|\,|\xi|}\right)\rho_2(|\xi|)\rho_3(|x|)$$

We also set:

$$\begin{aligned} p_\pm(t,h) = {} & \frac{i}{h}(|\partial_x\phi_\pm|^2 + V_1 - \xi^2)b_\pm(t,h) \\ & + (\partial_t b_\pm(t,h) + 2\partial_x b_\pm(t,h).\partial_x\phi_\pm + b_\pm(t,h)\Delta_x\phi_\pm \pm b_\pm(t,h)V_2) \\ & - ih^{N+1}\Delta_x b_\pm(t,h) \end{aligned}$$

as given by proposition 6.3.

Proposition 6.6. *The symbols $b_\pm$ and $p_\pm$ satisfy:*

(i) $\operatorname{supp} b_\pm \subset \Gamma_\pm(R_2, d_2, \pm\sigma_2)$ *and for $\pm t \geqslant 0$, $(x,\xi) \in \Gamma_\pm(R_2, d_2, \pm\sigma_2)$ and $\alpha, \beta \in \mathbb{N}^n$ we have:*
$$\left|\partial_x^\alpha \partial_\xi^\beta b(t,x,\xi,h)\right| \leqslant c_{\alpha,\beta}\,|t|^{N+(|\alpha|+|\beta|)(1-\rho)}\,\langle x\rangle^{-|\alpha|} \tag{6.17}$$

(ii) $\operatorname{supp} p_\pm \subset \Gamma_\pm(R_2, d_2, \pm\sigma_2)$ *and for $\pm t \geqslant 0$, $(x,\xi) \in \Gamma_\pm(R_2, d_2, \pm\sigma_2)$ and $\alpha, \beta \in \mathbb{N}^n$ we have:*
$$\left|\partial_x^\alpha \partial_\xi^\beta p_\pm(t,x,\xi,h)\right| \leqslant c_{\alpha,\beta}\,|t|^{N+(2+|\alpha|+|\beta|)(1-\rho)}\,\langle x\rangle^{-|\alpha|} \tag{6.18}$$

If furthermore $(x,\xi) \in \Gamma_\pm(R_3, d_3, \pm\sigma_3)$ then we have:

$$\left|\partial_x^\alpha \partial_\xi^\beta p_\pm(t,x,\xi,h)\right| \leqslant c_{\alpha,\beta} h^{N+1}\,|t|^{N+(2+|\alpha|+|\beta|)(1-\rho)}\,\langle x\rangle^{-2-|\alpha|} \tag{6.19}$$

Proof. (6.17) comes from (6.15). According to (6.13) and (6.14) we have:

$$p_\pm(t,x,\xi,h) = 2\partial_x\psi_\pm(x,\xi).\partial_x\phi_\pm(x,\xi)\sum_{j=0}^{N} a_{j,\pm}(t,x,\xi) - ih^{N+1}\Delta_x b_\pm(t,x,\xi,h)$$

so (6.18) is a consequence of (6.15) and (6.17). Finally, it remains to remark that for $\pm t \geqslant 0$ and $(x,\xi) \in \Gamma_\pm(R_3, d_3, \pm\sigma_3)$ we have $p_\pm(t,h) = -ih^{N+1}\Delta_x b_\pm(t,h)$ to get (6.19) from (6.17). □

Proposition 6.7. *Let $R_5 \in]R_3, R[$, $d_5 \in]d_3, d[$ and $\sigma_5 \in]\sigma_3, \sigma[$. There exists a symbol $e_\pm(h)$ of the form $e_\pm(h) = \sum_{j=0}^{N} h^j f_{j,\pm}$ with $f_{j,\pm} \in \mathcal{S}_{-j}$ and* $\operatorname{supp} f_{j,\pm} \subset \Gamma_\pm(R_5, d_5, \pm\sigma_5)$ *such that:*

$$I_h(b_\pm(0,h),\phi)I_h(e_{\nu,\pm}(h),\phi)^* = \omega_\pm(x,hD) + h^{N+1}Op_h(r_\pm(h))$$

where $r_\pm \in \mathcal{S}_{-N}$ uniformly in h.

Proof. This is lemma 4.5 in [Wan88]. Note that $b_\pm(0,h)$ is just $\psi_\pm$. □

Proposition 6.8. *For all $\delta \in \mathbb{R}$, there is $\nu \in \mathbb{N}$ such that for all $l \in \mathbb{R}$ and $\pm t \geqslant 0$ we have:*

$$\left\|\langle x\rangle^l I_h(b_\pm(t,h),\phi)U_0^h(t)I_h(e_\pm,\phi)^*\langle x\rangle^{-1-\nu-l}\right\| \leqslant c\,\langle t\rangle^{-\delta} \tag{6.20}$$

and:
$$\left\|\langle x\rangle^l I_h(p_\pm(t,h),\phi)U_0^h(t)I_h(e_\pm,\phi)^*\langle x\rangle^{-1-\nu-l}\right\| \leqslant ch^{N+1}\,\langle t\rangle^{-\delta} \tag{6.21}$$

Proof. For $u \in \mathcal{S}(\mathbb{R}^n)$ we have:

$$\begin{aligned}
&I_h(b_\pm(t,h),\phi_\pm)U_0^h(t)I_h(e_\pm(h),\phi_\pm)^* u(x)\\
&\qquad = \frac{1}{(2\pi h)^n}\int_y\int_\xi e^{\frac{i}{h}\zeta_\pm(t,x,y,\xi)} b_\pm(t,x,\xi,h)\overline{e_\pm(y,\xi,h)}u(y)\,d\xi\,dy
\end{aligned}$$

with $\zeta_\pm(t,x,y,\xi) = \phi_\pm(x,\xi) - \phi_\pm(y,\xi) - t\xi^2$. If R is large enough then for $(y,\xi) \in \operatorname{supp} e_\pm$ we have:

$$\left|\partial_\xi\phi_\pm(y,\xi) + 2t\xi\right| \geqslant \langle \partial_\xi\phi_\pm(y,\xi) + 2t\xi, \hat{y}\rangle \geqslant |y| - c\,|y|^{1-\rho} + 2\sigma_5\,|t|\,|\xi| \geqslant c_0(|y|+|t|) \tag{6.22}$$

for some $c_0 > 0$.

We consider the operator L such that for $u \in \mathcal{S}(\mathbb{R}^{2n})$:

$$Lu = ih\frac{(\partial_\xi\phi_\pm(y,\xi)+2t\xi).\partial_\xi u}{|\partial_\xi\phi_\pm(y,\xi)+2t\xi|^2}$$

Then we have:

$$L^* v = ih\,\mathrm{div}_\xi\,.\left(\frac{\partial_\xi\phi_\pm(y,\xi)+2t\xi}{|\partial_\xi\phi_\pm(y,\xi)+2t\xi|^2}\,v\right)$$

In particular $L\left(e^{-\frac{i}{h}(\phi_\pm(y,\xi)+t\xi^2}\right) = e^{-\frac{i}{h}(\phi_\pm(y,\xi)+t\xi^2)}$ so for $\nu \in \mathbb{N}$:

$$\begin{aligned}
&I_h(b_\pm(t,h),\phi_\pm)U_0^h(t)I_h(e_\pm(h),\phi_\pm)^* u(x)\\
&\quad = \frac{1}{(2\pi h)^n}\int_y\int_\xi e^{-\frac{i}{h}(\phi_\pm(y,\xi)+t\xi^2)}(L^*)^\nu\left(e^{\frac{i}{h}\phi_\pm(x,\xi)}b_\pm(t,x,\xi,h)\overline{e_\pm(y,\xi,h)}\right)u(y)\,d\xi\,dy
\end{aligned}$$

We can check by induction on $\nu \in \mathbb{N}$ that:

$$(L^*)^\nu\left(e^{\frac{i}{h}\phi_\pm(x,\xi)}b_\pm(t,x,\xi,h)\overline{e_\pm(y,\xi,h)}\right) = \sum_{j=1}^{J_\nu} e^{\frac{i}{h}\phi_\pm(x,\xi)} b^j_{\nu,\pm}(t,x,\xi,h)\overline{e^j_{\nu,\pm}(y,\xi,h)}$$

for some $J_\nu \in \mathbb{N}$ and for all $j \in [\![1,J_\nu]\!]$ we have:

$$\left|\partial_x^\alpha\partial_\xi^\beta b^j_{\nu,\pm}(t,x,\xi,h)\right| \leqslant c_{\alpha,\beta}\,|t|^{N-(|\alpha|+|\beta|)(1-\rho)-\rho\nu}\,\langle x\rangle^{\nu-|\alpha|}$$

and $e_0 \in \mathcal{S}_0$: Indeed, this is true for $\nu = 0$ by (6.17) and if this is true for some $\nu \in \mathbb{N}$ then for $j \in [\![1,J_\nu]\!]$ we have to compute:

$$\begin{aligned}
&ih\,\mathrm{div}_\xi\left(\frac{\partial_\xi\phi_\pm(y,\xi)+2t\xi}{|\partial_\xi\phi_\pm(y,\xi)+2t\xi|^2}e^{\frac{i}{h}\phi_\pm(x,\xi)}b^j_{\nu,\pm}(t,x,\xi,h)\overline{e^j_{\nu,\pm}(y,\xi,h)}\right)\\
&\quad = ih\,|\partial_\xi\phi_\pm(y,\xi)+2t\xi|^{-2}\times e^{\frac{i}{h}\phi_\pm(x,\xi)}\\
&\qquad \times\Bigg[(\Delta_\xi\phi_\pm(y,\xi)+2tn)b^j_{\nu,\pm}(t,x,\xi,h)\overline{e^j_{\nu,\pm}(y,\xi)}\\
&\qquad\qquad +2\frac{(\mathrm{Hess}_\xi\,\phi_\pm(y,\xi)+2tI_n).(\partial_\xi\phi_\pm(y,\xi)+2t\xi)^2}{|\partial_\xi\phi_\pm(y,\xi)+2t\xi|^2}b^j_{\nu,\pm}(t,x,\xi,h)\overline{e^j_{\nu,\pm}(y,\xi)}\\
&\qquad\qquad +\frac{i}{h}(\partial_\xi\phi_\pm(y,\xi)+2t\xi)\partial_\xi\phi_\pm(x,\xi)b^j_{\nu,\pm}(t,x,\xi,h)\overline{e^j_{\nu,\pm}(y,\xi)}\\
&\qquad\qquad +(\partial_\xi\phi_\pm(y,\xi)+2t\xi).\partial_\xi b^j_{\nu,\pm}(t,x,\xi,h)\,\overline{e^j_{\nu,\pm}(y,\xi)}\\
&\qquad\qquad +b^j_{\nu,\pm}(t,x,\xi,h)\,(\partial_\xi\phi_\pm(y,\xi)+2t\xi).\partial_\xi\overline{e^j_{\nu,\pm}(y,\xi)}\qquad\Bigg]
\end{aligned}$$

and check each term using (6.22). Note that the factor $\langle x\rangle^{\nu}$ in the estimate is due to the third term. We only gain a power $t^{-\rho\nu}$ at each iteration because of the fourth term and the fact that we have a bad estimate in t for the derivatives of $b_{\nu,\pm}$. Nonetheless, for all $\nu \in \mathbb{N}$ we get:

$$I_h(b_\pm(t,h),\phi_\pm)U_0^h(t)I_h(e_\pm(h),\phi_\pm)^* = \sum_{j=1}^{J_\nu} I_h(b^j_{\nu,\pm}(t,h),\phi_\pm)U_0^h(t)I_h(e^j_{\nu,\pm}(h),\phi_\pm)^* \quad (6.23)$$

For any $\nu \in \mathbb{N}$, the two operators $U_0^h(t)$ and $I_h(e_{\nu,\pm}(h),\phi_\pm)^*$ are uniformly bounded in t and h from $L^{2,1+l+\nu}$ into itself. The norm of $I_h(b_{\nu,\pm}(t,h),\phi_\pm)$ from $L^{2,1+l+\nu}$ to $L^{2,l}$ is estimated by a finite number of derivatives of $b^j_{\nu,\pm}$, say M (see [Wan88]). Then we have to choose ν such that $N + M(1-\rho) - \nu\rho \leqslant -\delta$ to obtain (6.20).

To prove (6.21) we introduce a function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(s) = 0$ if $s \leqslant \sigma_3$ and $\chi(s) = 1$ if $s \geqslant \sigma_4 \in]\sigma_3, \sigma_5[$. Then we write $p_{2,\pm}(t,x,\xi,h) = p_\pm(t,x,\xi,h)\chi\left(\pm\frac{\langle x,\xi\rangle}{|x||\xi|}\right)$ and $p_{1,\pm}(t,x,\xi,h) = p_\pm(t,x,\xi,h) - p_{2,\pm}(t,x,\xi,h)$. We have:

$$\left|\partial_x^\alpha \partial_\xi^\beta p_{2,\pm}(t,x,\xi,h)\right| \leqslant c_{\alpha,\beta} h^{N+1} |t|^{N+(2+|\alpha|+|\beta|)(1-\rho)} \langle x\rangle^{-2-|\alpha|}$$

The same argument as above proves (6.21) with $p_\pm$ replaced by $p_{2,\pm}$.

For $p_{1,\pm}$, we remark that for $(x,\xi) \in \operatorname{supp} p_{1,\pm} \subset \mathbb{R}^{2n} \setminus \Gamma_\pm(R_4,d_4,\pm\sigma_4)$ and $(y,\xi) \in \operatorname{supp} e_\pm \subset \Gamma_\pm(R_5,d_5,\pm\sigma_5)$ we have:

$$|\partial_\xi \zeta_\pm(x,y,\xi,t)| \geqslant c_0(|x|+|y|+|t|)$$

for some $c_0 > 0$. Indeed we have:

$$|\partial_\xi \zeta(x,y,\xi,t)| = |\partial_x \phi_\pm(x,\xi) - \partial_\xi \phi_\pm(y,\xi) - 2t\xi| \geqslant |x - (y+2t\xi)| - cR^{-\rho}$$

But $(y+2t\xi,\xi) \in \Gamma_\pm(R_4,d_4,\pm\sigma_4)$ so if $|x| \geqslant \gamma\,|y+2t\xi|$:

$$|x-(y+2t\xi)| \geqslant (1-\gamma^{-1})\,|x| \geqslant \frac{1-\gamma^{-1}}{2}(|x|+|y+2t\xi|) \geqslant c_0(|x|+|y|+|t|)$$

and if $|x| \leqslant |y+2t\xi|$:

$$\begin{aligned}|x-(y+2t\xi)| &\geqslant \left\langle x-(y+2t\xi), \mp\hat{\xi}\right\rangle = \frac{\pm 1}{|\xi|}(\langle y+2t\xi,\xi\rangle - \langle x,\xi\rangle)\\ &\geqslant (\sigma_5\,|y+2t\xi| - \sigma_4\,|x|) \geqslant (\sigma_5-\sigma_4)\,|y+2t\xi| \geqslant c_0(|x|+|y+2t\xi|)\\ &\geqslant c_0(|x|+|y|+|t|)\end{aligned}$$

Then we can do partial integrations with the operator $L = \frac{\partial_\xi\zeta.\partial_\xi}{|\partial_\xi\zeta|^2}$, each iteration giving a new power of h and $t^{-\rho}$. □

Corollary 6.9. *For all $\delta \in \mathbb{R}$, there is $\nu \in \mathbb{N}$ such that for all $l \in \mathbb{R}$ and $\pm t \geqslant 0$ we have:*

$$\left\|\langle x\rangle^l Op_h(\omega) I_h(b_\pm(t,h),\phi)U_0^h(t)I_h(e_\pm,\phi)^* \langle x\rangle^{-1-\nu-l}\right\| \leqslant ch^{N+1}\langle t\rangle^{-\delta} \quad (6.24)$$

Proof. The proof is the same as for (6.20) but instead of an estimate of $\left\|I_h(b^j_{\nu,\pm},\phi)\right\|$ we need an estimate of $\left\|Op_h(\omega)I_h(b^j_{\nu,\pm},\phi)\right\|$. According to lemma 4.4 in [Wan88] if we take R large enough, then the supports of $\omega(x,\partial_x\phi(x,\xi))$ and $b^j_{\nu,\pm}$ are disjoint, so this norm is only the norm of the rest given in proposition A.3 of [Wan88]. This rest is of order $O(h^{N+1})$ and the time dependance is given as for $\left\|I_h(b^j_{\nu,\pm},\phi)\right\|$ by a finite number of derivatives of $b^j_{\nu,\pm}$ so we conclude the same way. □

Now we can prove the main theorem of this section:

Proof of theorem 6.1. Let $\nu \in \mathbb{N}$ given by proposition 6.8 for $\delta = 2$. We prove the "+" case, and we omit the + subscript for ϕ, b, p and r. Let $t \geqslant 0$. According to (6.8) and proposition 6.3, we have:

$$U_h(t)I_h(b(0,h),\phi) = I_h(b(t,h),\phi)U_0^h(t) - \int_0^t U_h(t-s)I_h(p(s,h),\phi)U_0^h(s)\,ds$$

and then, by proposition 6.7:

$$\begin{aligned} U_h(t)Op_h(\omega_+) = h^{N+1}U_h(t)Op_h(r(h)) + I_h(b(t,h),\phi)U_0^h(t)I_h(e(h),\phi)^* \\ - \int_0^t U_h(t-s)I_h(p(s,h),\phi)U_0^h(s)I_h(e(h),\phi)^*\,ds \end{aligned}$$

For $\alpha > \frac{1}{2}$ and $\operatorname{Im} z > 0$, using $(H_h - z)^{-1} = \frac{i}{h}\int_0^\infty e^{\frac{it}{h}z}U_h(t)\,dt$ (see theorem 1.10 in [EN00]) gives:

$$\begin{aligned} &\langle x\rangle^{-\alpha} Op_h(\omega)(H_h - z)^{-1}Op_h(\omega_+)\langle x\rangle^{-\nu} \\ &\quad = h^{N+1}\langle x\rangle^{-\alpha} Op_h(\omega)(H_h - z)^{-1}Op_h(r(h))\langle x\rangle^{-\nu} \\ &\quad\quad + \frac{i}{h}\langle x\rangle^{-\alpha}\int_{t=0}^\infty e^{\frac{it}{h}z}Op_h(\omega)I_h(b(t,h),\phi)U_0^h(t)I_h(e(h),\phi)^*\langle x\rangle^{-\nu}\ dt \\ &\quad\quad - \langle x\rangle^{-\alpha} Op_h(\omega)\int_{s=0}^\infty e^{\frac{is}{h}z}(H_h - z)^{-1}I_h(p(s,h),\phi)U_0^h(s)I_h(e(h),\phi)^*\langle x\rangle^{-\nu}\ ds \end{aligned}$$

According to the uniform estimate for the resolvent (see [Roy]) the first term is $O(h^N)$. We use (6.24) and (6.21) for the second and third terms, which, after taking the limit $z \to E_h$ if $E_h \in \mathbb{R}$, proves (6.1). $\square$

Remark. To prove (6.2) we apply the same argument with:

$$(H_h^* - z)^{-1} = -\frac{i}{h}\int_{-\infty}^0 e^{-\frac{it}{h}(H_h^* - z)}\,dt$$

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Julien Royer
Laboratoire de mathématiques Jean Leray
2, rue de la Houssinière - BP 92208
F-44322 Nantes Cédex 3
France
julien.royer@univ-nantes.fr